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Critical Phenomena in Disordered Systems: A Renormalization-Group Approach

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A renormalization-group method is developed to study critical phenomena in a disordered system. The method bases its considerations on a generalized probability distribution which incorporates both thermodynamic and configurational averaging. Special attention is paid to the higher-order critical phenomena that occur at the percolation limit. A crossover scaling theory describes this region. The equation of the critical line is obtained for the Ising model.

The method of the renormalization group (RG)[1] has provided a very successful understanding of a variety of critical phenomena.[2] Recently, several studies have been made to obtain a similar understanding of critical phenomena in random spin systems.[3-6] In this Letter, I present another approach for the random systems. I base my considerations on a generalized probability distribution which incorporates both thermodynamic and configurational averaging. This allows me to discuss the higher-order critical phenomena which generally occur as the disorder parameter is increased to a critical value. The most familiar example of such a transition is the "percolation transition" that is obtained in the diluted Ising model, when the concentration of, say, "magnetic bonds" drops below the percolation concentration limit.[7,8]

Let us begin our considerations with a lattice of n-component classical spins $\bar{S}_i$ coupled by random exchange integrals. The system is described by the following Hamiltonian:

$$\mathcal{H}=\sum_{i,j} J_{ij}(1+\xi_i\xi_j)\bar{S}_i\cdot\bar{S}_j, \tag{1}$$

where the J_{ij}'s are short-ranged exchange integrals. The randomness is introduced through random variables ξ_i which have a given probability distribution $\Psi(\{\xi_i\})$. The molecular-field-theory type of arguments[9] show that the Curie temperature T_C of the system falls linearly with the characteristic mean-square width W of the probability distribution of the exchange integral, becoming zero as W approaches a critical value W_c.

In order to take into account the externally imposed randomness, we make a slight extension of the usual treatment of the RG theory. The thermodynamic average of a given quantity $A(\{\bar{S}_i\})$ for the random system is given by

$$\langle A\rangle=\int \prod_i d\xi_i\, U(\{\xi_i\})\int \prod_i d^3S_i\, A(\{\bar{S}_i\})\exp[-\beta\mathcal{H}(\{\bar{S}_i\},\{\xi_i\})+\beta\mathfrak{F}(\{\xi_i\})], \tag{2}$$

where $\mathfrak{F}(\{\xi_i\})$ denotes the free energy of the system in a given configuration. Thus the overall probability distribution describing thermodynamical and random averagings may be written as

$$P(\{S_i\},\{\xi_i\})=\exp[-H+\Phi(\{\xi_i\})+F(\{\xi_i\})], \tag{3}$$

where H, F, and Φ denote $\beta\mathcal{H}$, $\beta\mathfrak{F}$, and $\ln\Psi$, respectively. We now study RG transformations on P, but we also do a partial integration over the probability variables ξ_i. For this purpose, following Wil-

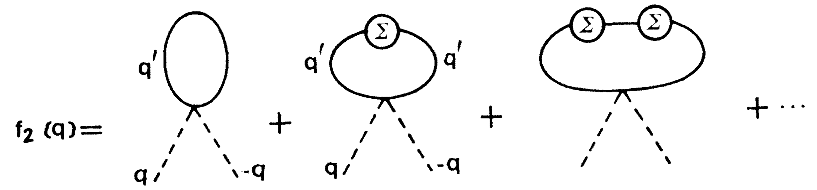

FIG. 1. The free-energy diagrams for the coefficient $f_2(q)$ defined in Eq. (6). The full lines represent spin propagators, the dotted lines represent the disorder variable $\xi(q)$, and $\Sigma(q)$ denotes the usual self-energy of the propagator.

son, we go over to the continuous-spin version of the above Hamiltonian. We take the variables ξ_i to be independently distributed, so that $\Phi(\{\xi_i\})=\sum_i\varphi(\xi_i)$. For the purpose of ϵ expansion, a suitable choice of $\varphi(\xi_i)$ may be taken as $-\xi_i^2/2W-\frac{1}{4}V\xi_i^4$. By taking the Fourier transforms, rescaling spin variables, etc., Eq. (3) can be cast in the form

$$\begin{aligned}P(\{\xi_q\},\{\sigma_q\})=N\exp[-\tfrac{1}{2}\sum_q\{\sum_\alpha(r+q^2)\sigma_\alpha(q)\sigma_\alpha(-q)+W^{-1}\xi(q)\xi(-q)\}\\ +4^{-d}\sum_{q_1,q_2,q_3}\{u\sum_{\alpha,\beta}\sigma_\alpha(q_1)\sigma_\alpha(q_2)\sigma_\beta(q_3)\sigma_\beta(-q_1-q_2-q_3)\\ +V\xi(q_1)\xi(q_2)\xi(q_3)\xi(-q_1-q_2-q_3)+2K(q_1-q_3)\sum_\alpha\sigma_\alpha(q_1)\sigma_\alpha(q_2)\\ \times\xi(q_3)\xi(q_3-q_1-q_2)\}+F(\{\xi_q\})].\end{aligned}\tag{4}$$

Simple considerations of diagrams for free energy show (see Fig. 1) that it is possible to write F as

$$F=L^df_0+\tfrac{1}{2}\sum_q f_2(q)\xi(q)\xi(-q)+(4L^d)^{-1}\sum f_4(q_1,q_2,q_3)\xi(q_1)\xi(q_2)\xi(q_3)\xi(-q_1-q_2-q_3)+\dots,\tag{5}$$

where f_0 is simply the free energy of the pure system, and f_2 and f_4 can be obtained diagrammatically and are in general singular functions of temperature. The coefficient $f_2(q)$ has a special role in our considerations and can easily be obtained from the diagrams of Fig. 1,

$$f_2(q)=\frac{1}{L^d}\sum_{q',\alpha}K(q-q')G_\alpha(q'),\tag{6}$$

where $G_\alpha(q)=\langle\sigma_\alpha(q)\sigma_\alpha(-q)\rangle$. Since $K(q)$ is a short-ranged interaction of the form $K-Aq^2$, we evaluate Eq. (6) as

$$f_2(q)=K[g_1(t)t^{1-\alpha}+g_0(t)+O(t^{2-\alpha})]-Aq^2[g_0(t)+g_2t^{1-\alpha}],\tag{7}$$

where $t=(T-T_{C_0})/T_{C_0}$, with T_{C_0} the Curie temperature of the pure system; g_0 and g_1 are analytic functions of t, determined by the pure system. Substituting Eq. (7) in Eq. (4), we find that the quadratic terms in ξ can be written as

$$\tfrac{1}{2}a\sum_q(s+q^2)\xi(q)\xi(-q),\tag{8}$$

and furthermore, s may be written as $[W^{-1}-W_c^{-1}(T)]/a$.

As may be seen from Eq. (7), the quantities a, s, and $W_c(T)$ are smoothly varying functions of t, although they do not have the usual Taylor expansion around $t=0$. The point about Eq. (8) is that the propagator for the random variable has become q dependent. By rescaling the random variable ξ to absorb the constant a, dropping the q dependence of K and f_4, etc., we are able to write Eq. (5) as

$$P=N\exp\left[-\left\{\sum_{\alpha=1}^{n+1}(r_\alpha+q^2)\sigma_\alpha(q)\sigma_\alpha(-q)+\frac{1}{4L^d}\sum_{\alpha,\beta}u_{\alpha\beta}\sigma_\alpha(q_1)\sigma_\beta(q_2)\sigma_\beta(q_3)\sigma_\beta(-q_1-q_2-q_3)\right\}\right],\tag{9}$$

where $\sigma_{n+1}(q)=\xi(q)$ and $r_{n+1}=s$. The notation $u_{\alpha\beta}$ is obvious from Eq. (4). The effective Hamiltonian of Eq. (9) has been extensively studied in various contexts.[5,10-12] The particular form of anisotropy that occurs in Eq. (9) has also been analyzed by Nelson, Kosterlitz, and Fisher.[12] A detailed analysis of the recursion relations and their fixed points may be found in Ref. 12.

We first note certain mathematical points about the various fixed points and consider the physical in-

TABLE I. Fixed points of the recursion relations of the Hamiltonian of Eq. (9). Here x is the real root of $9(4n^2+37n+121)x^3-6(2n^2+32n+211)x^2+(n^2+7n+478)x+6(n-10)=0$.

	r^*	s^*	u^*	K^*	V^*
I Gaussian	0	0	0	0	0
II n-anisotropic	$-\frac{\Lambda^2}{2}\frac{(n+2)}{n+8}\epsilon$	∞	$\frac{8\pi^2\epsilon}{n+8}$	0	0
III 1-anisotropic	∞	$-\frac{\Lambda^2}{2}\frac{\epsilon}{3}$	0	0	$\frac{8\pi^2\epsilon}{9}$
IV Decoupled $n-1$	$-\frac{\Lambda^2}{2}\frac{(n+2)}{n+8}\epsilon$	$-\frac{\Lambda^2}{2}\frac{\epsilon}{3}$	$\frac{8\pi^2\epsilon}{n+8}$	0	$\frac{8\pi^2\epsilon}{9}$
V Isotropic 1	$-\frac{\Lambda^2}{2}\frac{(n+3)}{n+9}\epsilon$	$-\frac{\Lambda^2}{2}\frac{(n+3)}{n+9}\epsilon$	$\frac{8\pi^2\epsilon}{n+9}$	$\frac{8\pi^2\epsilon}{n+9}$	$\frac{8\pi^2\epsilon}{n+9}$
VI Isotropic 2	$-\frac{\Lambda^2}{2}\frac{(n+3)}{n+9}\epsilon$	$-\frac{\Lambda^2}{2}\frac{(n+3)}{n+9}\epsilon$	$\{1+[1-(n+8)x^2]^{1/2}\}\frac{\pi^2\epsilon}{n+8}$	$\pi^2\epsilon x$	$[1+(1-9nx^2)^{1/2}]\frac{\pi^2\epsilon}{9}$

terpretation in the following paragraphs. There are six fixed points and their values are tabulated in Table I. The anisotropic fixed points II and III are the stable ones. When $r<s$, the n-anisotropic fixed point II is stable, while when $r>s$, the 1-anisotropic fixed point III is stable. The latter fixed point pertains to external probability distribution only and is of no interest to our problem. The n-anisotropic fixed point is simply an n-Heisenberg fixed point. When $r\approx s$, the behavior is dominated by isotropic fixed points V and VI. Both these fixed points are doubly unstable (i.e., we must fix both temperature and disorder parameter to be at these fixed points). The analysis of Ref. 12 shows that when $1+n<n^x(d)\simeq(4+3.176\epsilon)/(1+1.2948\epsilon)$ the isotropic fixed point V, which is $(n+1)$-Heisenberg, is the more stable one. This fixed point describes a bicritical behavior. The fixed point VI becomes stable when $n^x(d)<n+1<11$, and it describes a tetracritical behavior.

I shall first discuss the bicritical situation. In Fig. 2, one can identify the critical line with the n-anisotropic fixed point. Thus, the asymptotic critical behavior along this line is that of a pure n-Heisenberg system as long as we are away from the point $W=W_c(0)$. The point $W=W_c(0)$ should be identified with the bicritical fixed point.[13] The second critical fluctuation occurring here may be viewed as a fluctuation in disorder occurring in the vicinity of the percolation transition. The critical behavior of the system near the bicritical point will be like that of an $(n+1)$-isotropic Heisenberg point with the characteristic crossover exponent $\varphi=1+[(n+1)/2(n+8)]\epsilon$.[13] Following Riedel and Wegner,[13,14] one should discuss this situation in terms of effective critical indices. The critical indices at a point in W-T plane assume intermediate values lying between the two asymptotic values, depending upon the distances of the given point from the bicritical fixed point and the critical line.

In order to make the above remarks concrete, we discuss the simplest case of $n=1$. The linear scaling fields at the bicritical point are

$$\mu_1(T,W)=r_1+s-r^*,$$
$$\mu_2(T,W)=r_1-s-r^*. \qquad (10)$$

These fields give the directions in the W-T plane along which the thermodynamic quantities have simple power-law-like behavior. In order to express these variables in terms of T and W, it should be noted that the bicritical point occurs at $T=0$. For the Ising model, at $T=0$ the proper

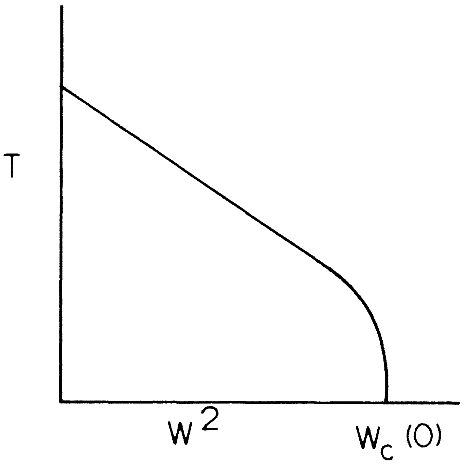

FIG. 2. Variation of the Curie temperature with the disorder parameter W^2. In the continuous-spin model the variable on the x axis is changed to W.

expansion variable is not T but $\exp(-aT_C/T)$, where $a \sim O(1)$. This may easily be verified from the low-temperature molecular-field–theoretic expression or otherwise.[8] Thus, the quantities $r(T)$, $W_c(T)$, etc. have the following expansions:

$$r(T) = j\exp(-aT_C/T),$$
$$W_c(T) = W_c(0)\{1 - f\exp(-aT_{C_0}/T)\}. \tag{11}$$

Substituting these in Eq. (10), one obtains the equation of the critical line as

$$\exp[-aT_{C_0}/T_C(W)] = B[W_c(0) - W] - B'[W_c(0) - W]^{\varphi}. \tag{12}$$

This analytical dependence of the critical line on the disorder variable is the same as was obtained for the annealed impurity case by Rappaport.[7] The scaling fields derived here are similar to those proposed by Stauffer.[8] For $n > 1$, the scaling fields will have different temperature dependences due to the presence of spin waves. This matter is under investigation.[15]

My results for exponents are qualitatively consistent with Rappaport's series-analysis results for the diluted Ising model. He found that the susceptibility exponent increases from the pure-system value to about the Fisher renormalized value as the concentration of magnetic bonds is decreased to the percolation limit. In my analysis the exponents cross over from n-Heisenberg-like to $(n+1)$-Heisenberg-like, which is a comparatively smaller variation. The present theory is also consistent with the RG calculations on the planar Ising model,[3] and with the arguments of Domb,[7] in which one finds no change in critical indices. My results, however, are not in accord with the other ϵ-expansion theories,[3-5] in which one finds that for $1 < n \lesssim 4$, the critical indices are different from those of the pure system for any value W of disorder. The discrepancies between these results are not understood at the moment.

The existence of the tetracritical fixed point for $n > n^x(d) - 1$ implies a more complex critical behavior, in which, presumably, ferromagnetic ordering competes with a spin-glass–type of ordering. This point is under further investigation.[15]

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