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Statistical Analysis of Feynman Diagrams

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The methods of statistical mechanics are used to evaluate sums of many-vertex Feynman graphs in a simple quantum-field-theory model.

There has been much discussion during the past decade of the function $F_1(\alpha)$, the coefficient of the logarithmically divergent contribution of the one-fermion-loop Feynman diagrams to the photon propagator in massless quantum electrodynamics. It has been shown that the zeros of $F_1(\alpha)$ are *essential* zeros (points where F_1 vanishes more rapidly than any power of x)[1] and it is conjectured that one of these zeros is the actual physical dimensionless charge on the electron ($\sim \frac{1}{137}$). $F_1(\alpha)$ may be calculated in perturbation theory although the calculation is difficult and only the first three terms are known[2]:

$$F_1(\alpha)=\tfrac{2}{3}\alpha+\alpha^2-\tfrac{1}{4}\alpha^3+\dots . \qquad (1)$$

However, the location of an essential zero of $F_1(\alpha)$ is fixed by the radius of convergence of this series; this, in turn, is determined not by the leading orders of perturbation theory, but rather by the nth term in the limit as $n\to\infty$. Thus, what is really needed is a method for evaluating sums of Feynman diagrams as n, the number of vertices, approaches ∞.

In this paper we introduce a novel statistical method for analyzing the behavior of a Feynman diagrammatic perturbation theory as the order of the perturbation theory gets large. We examine the distribution of n-vertex Feynman graphs, identify that class of graphs which contributes most to the perturbation expansion when n is large, and thereby determine the large-order behavior of the perturbation theory. Although we present these ideas in the context of a simple quantum-field-theory model, we hope that ultimately they will be useful for understanding such problems as the perturbative structure of $F_1(\alpha)$.

We have recently investigated the large-order behavior of the perturbation expansions for some quantum mechanical systems.[3-5] In all cases we have found that in large order, the structure of perturbation theory becomes extremely simple. For example, the ground-state energy $E(\lambda)$ of the generalized quantum anharmonic oscillator, defined by

$$\left[-\frac{d^2}{dx^2}+\frac{x^2}{4}+\lambda\,\frac{x^{2K}}{2^K}-E(\lambda)\right]y(x)=0,\qquad y(\pm\infty)=0, \qquad (2)$$

has a perturbation expansion of the form

$$E(\lambda)\sim\tfrac{1}{2}-\sum_{n=1}^{\infty}A_n(-\lambda)^n \quad (\lambda\to 0^+). \qquad (3)$$

We have shown that, apart from an overall multiplicative constant and algebraic dependence on n,[3]

$$A_n \sim 2^{-n}[n(K-1)]!\left(\frac{\Gamma(2K/(K-1))}{\Gamma^2(K/(K-1))}\right)^{n(K-1)} \quad (n\to\infty). \quad (4)$$

In this paper we are only concerned with the factorial and power growth of A_n. The radius of convergence of a perturbation series is not sensitive to algebraic dependences so they are ignored here.

Unfortunately, the behavior of A_n in (4) was derived from a semiclassical analysis of the differential equation (2) so the methods that were used are not applicable to the diagrammatic expansions of quantum field theory. Nevertheless, the simplicity of this result has led us to believe that it is possible to deduce the large-order behavior of a diagrammatic perturbation theory by direct analysis of the diagrams.

We are proposing new and intuitive methods; it is our purpose here to show that they actually work for a problem whose solution is already known. Thus, we will show how to rederive Eq. (4).

The perturbation series for E in (3) has a diagrammatic expansion because E is the ground-state energy of a φ^{2K} field theory in one-dimensional space-time whose Hamiltonian is

$$H=\tfrac{1}{2}\varphi^2+\tfrac{1}{2}\dot{\varphi}^2+\lambda\varphi^{2K},$$

and whose equal-time commutation relations are

$$[\varphi(t),\dot{\varphi}(t')]|_{t=t'}=1.$$

The ground-state energy E satisfies the equation

$$H|\text{g.s.}\rangle = E(\lambda)|\text{g.s.}\rangle .$$

A_n, the nth term in the perturbation expansion for $E(\lambda)$, is equal to the sum of all connected vacuum diagrams having n vertices.

The Feynman rules for computing A_n in momentum space are $(2K)!$ for a vertex, $1/(E^2-1+i\epsilon)$ for a propagator, and $\int(i/2\pi)dE$ for each loop integration. In coordinate space the rules are $\frac{1}{2}\exp(-|x_i-x_j|)$ for a line connecting the ith and jth vertices, and we must integrate over all but one vertex:

$$\iint\ldots\int dx_1 dx_2 dx_3 \ldots dx_{n-1}.$$

(The last integration $\int dx_n$ gives a volume divergence which reflects the time translation invariance of bubble diagrams.)

After calculating several hundred bubble graphs we find strong indications that the calculation of A_n for large n is indeed a problem in statistical mechanics. The factorial dependence $[n(K-1)]!$ in (4) arises from the overall *number* of graphs and not from some special topological class of graphs (like tower diagrams, for example). In any number of space-time dimensions the number of vacuum graphs for a φ^{2K} field theory is

$$[n(K-1)]![2K/(K-1)]^{Kn}(K-1)^n, \quad (5)$$

apart from algebraic dependences on n.

Individual diagrams grow no faster than c^n, where c is a constant. It is possible to define the value of an "average" graph by dividing the desired answer in (4) by the number of graphs in (3). For example, when $K=2$, the average graph behaves like $(\frac{3}{16})^n$. Some graphs grow faster than the average graph and some grow slower but average graphs dominate the perturbation expansion as $n\to\infty$ because there are more of them; the values of all n-vertex graphs have a Gaussian-like distribution which is sharply peaked at the value of the average graph.[6]

We briefly describe the calculational procedure:

(1) We work in coordinate space and number the vertices so that $x_i<x_j$ when $i<j$. Ordering the vertices in this way is not necessary but it simplifies the presentation because the propagator becomes the simple exponential $\frac{1}{2}\exp(x_i-x_j)$.[7]

(2) We take the number of vertices large: $n\gg 1$. Then we group the vertices into N boxes, placing n/N vertices in each box, as shown in Fig. 1. We choose $N\gg 1$, but $N^2\ll n$. It is not necessary to partition the vertices so that there

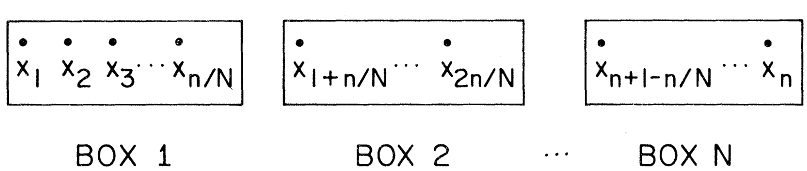

FIG. 1. Partitioning of vertices into boxes. We group the vertices into N boxes, placing n/N vertices in each box. The vertices are labeled $x_1\ldots x_n$.

are equal numbers in every box. The nonequipartition case gives identical results and is described in Ref. 6.

(3) We view these boxes as having an unspecified internal structure but we do specify how each box is (externally) joined to other boxes. The number of lines connecting the ith and jth box, where $1 \leq i,j \leq N$, is given by the connectivity matrix $F(i,j)$, which is symmetric and has positive integer entries. $F(i,i)$ is twice the number of internal lines in box i. Since there are n/N vertices per box, $F(i,j)$ must satisfy the constraint equation

$$\sum_{j=1}^{N} F(i,j) = 2Kn/N \tag{6}$$

for all i.

(4) We have organized all diagrams into classes, with each class specified by the matrix $(F(i,j)$. We must therefore determine which of these classes contributes most to A_n in (3) and (4). The contribution of the class associated with the matrix $F(i,j)$ is equal to the number of diagrams in that class multiplied by the value of any one such diagram. (All diagrams in any class have the same value because we disregard the internal structure of boxes.)

A counting argument shows that there are approximately

$$[(2K)!]^{-n}[(n/N)!]^N \left(\prod_{i,j=1}^{N} F(i,j)! \right)^{-1/2} \tag{7}$$

diagrams in the class $F(i,j)$.

To evaluate a Feynman integral for a diagram in class F we define the function

$$G(k) = \sum_{i \leq k} \sum_{j \geq k} F(i,j),$$

which is the number of lines one must cut to sever the diagram at the kth box. Using the coordinate-space Feynman rules, the value of a diagram is simply

$$[(2K)!]^n 2^{-Kn} \prod_{k=1}^{N} [G(k)]^{-n/N}. \tag{8}$$

The contribution of class $F(i,j)$ to A_n is just the product of (7) and (8).

(5) Next, we pass to the continuum limit by introducing the variables $x = i/N$, $y = j/N$, $z = k/N$, and $F(i,j) = (2Kn/N^2) f(x,y)$. $f(x,y)$ is a symmetric function whose arguments range from 0 to 1. In terms of these variables, the constraint equation (6) reads

$$\int_0^1 dx\, f(x,y) = 1. \tag{9}$$

Using the Stirling formula $Q! \sim Q^Q e^{-Q}$, where we have discarded algebraic factors, we obtain an expression for A_n:

$$A_n \sim e^{-nK}(nK)^{n(K-1)} 2^{-n} \sum e^{-nW[F]}, \tag{10}$$

where the sum is over all classes $F(i,j)$ and

$$W[F] = \int_0^1 dx \ln \int_0^z dx \int_z^1 dy\, f(x,y) + K \int_0^1 dx \int_0^1 dy\, f(x,y) \ln f(x,y). \tag{11}$$

Observe that all reference to N has disappeared.

(6) When n is large, we use Laplace's method[8] to expand the functional integral in (10) [again, the sum is over all classes $F(i,j)$] and obtain

$$\sum e^{-nW[F]} \sim e^{-nW[F_0]},$$

apart from algebraic factors. The dominant contribution to A_n is made by that class F_0 which minimizes $W[F]$ in (11). To minimize (11) subject to the constraint in (9) we introduce a Lagrange multiplier $\lambda(y)$ and take variational derivatives with respect to λ and f. We obtain an integral equation for $f_0(x,y)$:

$$f_0(s,t) = [f_0(s,s) f_0(t,t)]^{1/2} \exp[-(1/2K) | \int_s^t dz / \int_0^z dx \int_z^1 dy\, f_0(x,y) |]. \tag{12}$$

(7) The simultaneous integral equations (9) and (12) have an exact closed-form solution:

$$f_0(x,y) = \begin{cases} -A'(x)B'(y), & x<y, \\ -A'(y)B'(x), & x>y, \end{cases} \tag{13}$$

where $A(0)=0$, $B(1)=0$,

$$A(x)^{(K-1)/K}+B(x)^{(K-1)/K}=C,$$

$$A'(x)B(x)-B'(x)A(x)=1,$$

$$\ln C=-\frac{K-1}{2K}\ln\left(\frac{K}{K-1}\frac{[\Gamma(K/(K-1))]^2}{\Gamma(2K/(K-1))}\right). \qquad (14)$$

(8) Substituting (13) and (14) back into the expression for $W[F]$ and integrating by parts gives

$$W[F_0]=-1-2K\ln C,$$

and substituting this result into (10) recovers the formula in (4) for the behavior of A_n when n is large.

We are currently working on extending these methods to higher-dimensional field theories and hope to use them eventually to obtain the radius of convergence of $F_1(\alpha)$.

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[6]Plots of these distributions are given for various values n by C. M. Bender and T. T. Wu, to be published. We thank Mr. Robert Keener for his assistance in preparing these plots.

[7]A careful treatment of propagators which are not simple exponentials is given in Ref. 6.

[8]The use of Laplace's method here emphasizes that the behavior of perturbation theory in large order reflects the semiclassical content of the theory. A brief discussion of this point is given in Ref. 1.

Phase Transitions in the Quantum Heisenberg Model

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We rigorously prove that in three or more dimensions, the nearest-neighbor, simple-cubic, ferromagnetic, quantum Heisenberg model of spin S $(=\frac{1}{2},1,\cdots)$ has a phase transition at nonzero temperature.

The quantum Heisenberg model represents one of the simplest models in which ferromagnetic behavior occurs. From a variety of intuitions including high-temperature expansions[1] and spin-wave approximations,[2] it has been believed for many years that the three-dimensional model has a first-order phase transition in the magnetic field at sufficiently low temperatures. Despite its obvious physical interest, a rigorous proof of this has not been available—the only previous rigorous results on phase transitions in quantum lattice systems are proofs of the absence of phase