Tokamak Heat Transport Due to Tearing Modes*

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(Received 26 January 1976)

Magnetic field perturbations accompanying the tearing instability allow rapid heat transport across regions enclosing mode-rational equilibrium flux surfaces. The electron-energy transport equation for this process is derived, under the assumptions of cylindrical geometry, incompressible flow, and relatively weak density gradients. For the thermoelectric instability predicted by kinetic theory, a quasilinear description of local flattening of the temperatures profile is obtained. The self-consistent field perturbation is shown to be quite weak, $\delta B/B_0 \sim 10^{-4}$; yet it yields a localized heat flux far exceeding pseudoclassical estimates.

Soft x-ray measurements on tokamaks have disclosed rapid radial heat transport, associated with small-amplitude magnetic field perturbations.[1] The frequency of the observed small-amplitude oscillations is close to ω_*, the electron diamagnetic frequency. Thermal transport across the $q=m$ surface (here q is the Kruskal-Shafranov safety factor and $m=1,2,3$ is the poloidal mode number) is experimentally indicated by a flattening of the electron temperature profile over a small annular region, enclosing the $q=m$ surface. The temperature perturbation appears nearly uniform over the equilibrium flux surface.

Under some conditions the oscillations are followed by abrupt disruption of the plasma column —a process which shall not be considered here. We restrict our attention to the relatively mild perturbation in T_e, which should be amenable to a quasilinear treatment. Because the $m=1$ case is more complicated both experimentally and theoretically, we also restrict our attention to modes with $m \gtrsim 2$. We use a cylindrical geometry, and, noting that

$$n'/n \ll T_e'/T_e$$

(here n is the density and the primes denote radial derivatives) at radii of interest in most of the relevant experiments, we neglect the density gradient for simplicity. With these assumptions, and using the results of a previous study of linear tearing instabilities,[2] we derive a self-consistent, quasilinear description of localized flattening of the temperature profile.

We begin with the exact electron energy conservation law[3]:

$$(\partial/\partial t)(\tfrac{3}{2}nT) + \nabla\cdot(\tfrac{3}{2}nT\vec{v}) + nT\nabla\cdot\vec{v} + (\overleftrightarrow{\pi}\cdot\nabla)\cdot\vec{v} + \nabla\cdot\vec{q} = Q\,, \qquad (1)$$

where n and T are the electron density and temperature, respectively, $\vec{v}$ is the mean electron flow velocity, $\overleftrightarrow{\pi}$ is the stress tensor (in the sense of Braginskii[3]), $\vec{q}$ is the electron heat flux, and Q is the collisional energy exchange between electrons and ions. The terms involving $\overleftrightarrow{\pi}$, $\vec{q}_\perp$, and Q (proportional to m_e/m_i, or to the square of the gyroradius) are easily retained, but their effect turns out to be negligibly small compared to the transport associated with $q_\parallel = \vec{B}\cdot\vec{q}/B$ and $\vec{v}$. This is because, in long-mean-free-path regimes of interest, tearing instabilities allow radial heat transport *along* the magnetic field which easily dominates neoclassical, or even pseudoclassical, effects. The omission of $\overleftrightarrow{\pi}$, $\vec{q}_\perp$, and Q, which can be rigorously justified only *a posteriori* [cf. Eq. (20)], is closely analogous to the neglect, in classical tearing instability theory,[2,4] of terms describing resistive diffusion of the equilibrium magnetic field. We also assume incompressible flow. Then Eq. (1) reduces to

$$\tfrac{3}{2}n_0(\partial T/\partial t) + \tfrac{3}{2}n_0\vec{v}\cdot\nabla T + \vec{B}\cdot\nabla(q_\parallel/B) = 0. \qquad (2)$$

Here we have replaced n by its equilibrium value, n_0, because our neglect of ∇n and the incompressibility assumption imply that the density is unperturbed. In the same approximation, $\vec{v}\cdot\nabla T = \vec{v}_E\cdot\nabla T$, where $\vec{v}_E$ is the electric drift; the diamagnetic drift does not enter Eq. (2).

The parallel heat flux is assumed to obey the transport equation,[2]

$$q_\parallel = -L_{12}[eE_\parallel + n^{-1}\nabla_\parallel p] - L_{22}\nabla_\parallel T\,, \qquad (3)$$

where $p=nT$, $E_\parallel$ is the perturbed parallel electric field, and the transport coefficients L_{mn} may be written as

$$L_{mn} = (p\tau/m_e)\lambda_{mn}(\omega\tau). \qquad (4)$$

Here τ is the electron collision time of Braginskii and the λ_{mn} depend upon the wave frequency,

ω, as shown in Ref. 2.

In cylindrical coordinates (r,θ,z), with $\vec{B}_0 = \hat{\theta}B_{\theta 0}+\hat{z}B_{z0}$, and $\vec{B}_0\cdot\nabla T_0 = 0 = \vec{v}_{E0}$, the linearization of Eqs. (2) and (3) is straightforward. We find that the linear temperature perturbation, $T_1(r,\theta,z)$, is given by

$$T_1 = -\xi T_0' + \tfrac{2}{3}(k_\parallel/\omega n_0)q_\parallel. \quad (5)$$

Substituting this expression into Eq. (3), we obtain

$$hq_\parallel = -(b - ik_\parallel\xi)L_2T_0' - L_{12}eE_\parallel. \quad (6)$$

Here we use the notations

$$\xi \equiv iv_{Er}/\omega, \quad b\equiv B_{1r}/B_0, \quad L_2 = L_{12}+L_{22},$$

$$k_\parallel = (m/r)B_{\theta 0} + (l/R)B_{z0}, \quad h\equiv 1+\tfrac{2}{3}ik_\parallel^2L_2/\omega n_0.$$

The prime denotes a radial derivative. To complete the linear description, we use the radial component of Faraday's law, together with the tokamak ordering $B_\theta/B_z \ll 1$ and the identity $\vec{E} = -\vec{v}_E\times\vec{B}+\vec{E}_\parallel$, to obtain the familiar relation[4]

$$E_\parallel = (\omega rB_0/mc)(b - ik_\parallel\xi), \quad (7)$$

where m is the poloidal mode number.

A quasilinear description is obtained from the second-order version of Eq. (2), averaged in θ. Treating $\partial T_0/\partial t$ as second order, we have

$$\frac{3}{2}n_0\frac{\partial\langle T_0\rangle}{\partial t}+\frac{1}{r}\frac{1}{\partial r}r[\tfrac{3}{2}\langle v_{Er}T_1\rangle + \langle bq_\parallel\rangle] = 0, \quad (8)$$

where

$$\langle f\rangle = \oint (d\theta/2\pi)f,$$

and we have noted that, for any vector $\vec{A}$, $\langle\nabla\cdot A\rangle = r^{-1}(r\langle A_r\rangle)'$. According to Eq. (5),

$$\tfrac{3}{2}n_0\langle v_{Er}T_1\rangle + \langle bq_\parallel\rangle = \langle(b - ik_\parallel\xi)q_\parallel\rangle - \tfrac{1}{2}T_0'|\xi|^2\,\mathrm{Im}\omega,$$

where Eqs. (6) and (7) provide

$$(b - ik_\parallel\xi)q_\parallel = -h^{-1}(b - ik_\parallel\xi)^2 \times[L_2T' + (e\omega rB_0/mc)L_{12}]. \quad (9)$$

Before evaluating the θ average of Eq. (9), we specialize to the "thermoelectric" instability which was emphasized in Ref. 2. In the parameter range of interest, this mode is characterized by

$$\omega_r \equiv \mathrm{Re}\omega \simeq \omega_{e*}+0.8\omega_{T*}, \quad (10)$$

$$\gamma \equiv \mathrm{Im}\omega = (0.43)(\omega_{T*}\tau)\omega_r, \quad (11)$$

where

$$\omega_{T*} \equiv -(c/eB)(m/r)T_e',$$

$$\omega_{e*} = -(c/eB)(m/r)P_e'/n.$$

For typical tokamak parameters ($n\simeq 10^{13}$ cm^{-3}, $\tau^{-1}\simeq 10^4$ sec^{-1}), $\gamma/\omega_r \sim 10^{-1}$, so we approximate for $\gamma/\omega_r \ll 1$. Then the transport coefficients in Eq. (9) are nearly real, and

$$\langle(b - ik_\parallel\xi)q_\parallel\rangle \cong -(|b - ik_\parallel\xi|^2/|h|^2) \times(L_2T' + L_{12}e\omega_r r/m_e).$$

After substituting this expression into Eq. (8), and using Eq. (10), we obtain

$$\partial T/\partial t = r^{-1}\partial(r\kappa_* T')/\partial r, \quad (12)$$

where (using the λ_{mn} from Ref. 2)

$$\kappa_* = (T\tau/m_e)|b_*|^2/|h|^2,$$

with

$$b_* = b - ik_\parallel\xi.$$

The quasilinear description is completed by assuming

$$\frac{\partial}{\partial t}|b_*|^2 = 2\gamma|b_*|^2, \quad (13)$$

where γ is given by (11).

It is important to notice that the quantity b_*, unlike b, becomes small outside the tearing layer. For this reason, and because of the $|h|^{-2}$ factor, the thermal conductivity is localized to a narrow annular region enclosing the $k_\parallel = 0$ surface. Within this region, however, we shall see that κ_* is enormous, even for very small magnetic perturbations.

The spatial structure of κ_* may be represented, approximately, by[2]

$$\kappa_* \propto (1+x^4/\delta^4)^{-1}\exp(-x^2/\lambda^2), \quad (14)$$

where

$$x = r - r_0,$$

$$\lambda = (\Delta'/2)^{-1/3}(k_\parallel'a)^{-2/3}(\omega_*\tau_A)^{2/3}, \quad (15)$$

$$\delta = (k_\parallel')^{-1}\omega_*^{1/2}(m_e/T_e)^{1/2}\tau^{-1/2}. \quad (16)$$

Here r_0 is the radius of the mode-rational surface, $k_\parallel(r_0) = 0$, $\Delta' = [b(r_0+\lambda) - b(r_0-\lambda)]/b(r_0)$, and τ_A is the Alfvén time. For consistency with the experimental observations, both λ and δ should be of the order of 1 cm. We evaluate these lengths for the case of a locally flattened temperature gradient; to keep ω_* nonzero, the small density gradient term in ω_* is retained, using the estimate $n' \simeq n/5a$. Then, with $B \simeq 10^4$ G, $n = 10^{13}$ cm^{-3}, $T \simeq 10^3$ eV, $a \simeq 10$ cm, $r_0 \simeq 5$ cm, $m = 2$ and $\Delta' a = 20$,[5] we find from Eqs. (15) and (16) that both λ and δ have the desired magnitudes, provided that

$$k_\parallel' aR = (mRB_\theta / rB_z)' a \approx \tfrac{1}{25}. \qquad (17)$$

This value is quite small, and yet not inconsistent with some estimated tokamak current profiles.[5] Our theory thus predicts that anomalous thermal transport occurs over observably wide regions only if the shear is locally small.

We now consider the solution of our quasilinear equations. The growth rate of the linear mode depends on the temperature gradient, T'. As the mode grows, κ_* increases and T' decreases, until the mode saturates by locally flattening the temperature profile. We can calculate the saturation amplitude quite simply, using a conservation law which was derived for analogous quasilinear equations by Wong.[6] From Eq. (12) we obtain

$$\int dr\, rT \frac{\partial T}{\partial t} \simeq -\int dr\, r \frac{T\tau}{m_e} \frac{T'^2 |b_*|^2}{|h|^2},$$

where the integral extends over the larger of the two radial distances, λ or δ, and we have noted that $|b_*|/|h|$ becomes small at the integration limits. We see from Eqs. (11) and (13) that the integrand on the right-hand side is proportional to $\partial |b_*|^2/\partial t$. Thus, neglecting the time variation of $|h|$, we find

$$\int dr\, r(\partial T^2/\partial t + \alpha\, \partial |b_*|^2/\partial t) = 0, \qquad (18)$$

where

$$\alpha^{-1} = (cm/eBr)^2 m_e |h|^2 / T_e.$$

Equation (18) has been found useful in numerical studies of the quasilinear system,[7] but here we use it only to estimate the saturation amplitude of $|b|$. Evidently, $|b_*|^2 \sim \alpha^{-1} \int dr\, r \Delta T^2$, where

$$\Delta T^2 \equiv T^2(t=\infty) - T^2(t=0),$$
$$\sim 2(\lambda/a) T^2(t=0),$$

assuming that an initial, locally linear, temperature profile is flattened in a layer of width λ. Then

$$|b| \sim |b_*| \leq (m\rho_e/r)(\lambda/a)^{1/2}, \qquad (19)$$

where ρ_e is the electron gyroradius. For the parameters given above, this yields $|B_{1r}/B_0| \simeq 3 \times 10^{-4}$. Even at this small amplitude, the local thermal conductivity is very large; from the definition, and using the tokamak parameters listed previously, we find that Eq. (19) yields

$$\kappa_* \simeq 10^5 \kappa_{\mathrm{NC}}, \qquad (20)$$

where κ_{NC} is the corresponding coefficient obtained from neoclassical theory.

In order that our quasilinear description be valid, mode-mode coupling should be weak when the mode saturates. Mode-mode coupling becomes significant when the island width is comparable to λ[8]; this criterion is only marginally satisfied at the saturation amplitude of Eq. (19), but should be well satisfied during most of the quasilinear evolution.

We note that because κ_* is so large, the requirement of small shear can be considerably relaxed. If we consider that κ_* dominates κ_{NC} over a width of several tearing layers (which is of course small compared to the plasma radius), we can allow smaller λ and δ, hence larger $k_\parallel'$.

We wish to thank H. V. Wong for helpful discussions.

*This work was supported by U. S. Energy Research and Development Administration under Contract No. E-(40-1)-4478.

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