PHYSICAL REVIEW LETTERS

VOLUME 36 | 26 APRIL 1976 | NUMBER 17

Particle Creation by Gravitational Fields

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(Received 26 January 1976)

This paper presents a new and generally covariant approach to quantum field theory in curved space-time. Previously unknown expressions are obtained for particle creation amplitudes in oscillating gravitational fields. These expressions illuminate the physical origin of the particle creation.

Recent work[1-7] on particle creation by gravitational fields has raised, but not answered, the question, "To what extent is it possible to make sense of the occupation-number formalism in a curved space-time?" By analogy with the behavior of charged particles in external electromagnetic fields, a gravitational field will start spontaneously creating particles of rest mass m when the space-time radius of curvature r becomes comparable with the Compton wavelength[1] $\hbar/mc$ (the term "radius of curvature" is used loosely to denote any length scale associated with the gravitational field). When $r > \hbar/mc$, the creation rate is damped by a factor $\exp(\sim mcr/-\hbar)$. Thus one expects the occupation-number formalism to make sense when $r \gg \hbar/mc$ but to break down when $r \sim \hbar/mc$.

For massless particles, there is a satisfactory asymptotic formalism.[1,2] For massive particles, no covariant answer is known (even in the static case, there are severe problems[3]).

In this paper, the occupation-number formalism is introduced by replacing the plane waves used in flat space-time by JWKB waves. For the JWKB waves there is a natural distinction between positive and negative frequency, a distinction which leads to generally covariant (but inexact) definitions of vacuum states and of particle creation and annihilation operators in weak-field regions of space-time. The particle concept implied is the classical one and the ambiguity in the definition of the vacuum state corresponds precisely to the physical uncertainty caused by particle creation. Further, by examining the way in which the JWKB approximation breaks down as r decreases, it is possible to extend some of the techniques of perturbation theory which are familiar from flat space-time. This is illustrated by a computation of the threshold amplitude for pair production in a weak, but highly oscillating, gravitational field.

Consider a massive Hermitian scalar field operator Ψ satisfying the Klein-Gordon equation (hereafter, we take $c = 1$)

$$\hbar^2 \Box \Psi + m^2 \Psi = 0, \tag{1}$$

in a curved space-time M. Suppose that σ is some Cauchy hypersurface[8] and that V is some region of space-time containing σ. If the radius of curvature in V is very much greater than the Compton wavelength $\hbar/m$, then the classical Klein-Gordon equation can be solved in V in the JWKB approximation (that is, ignoring terms of order $\hbar^2$) by $\varphi = \rho \exp(iS/\hbar)$. Here S satisfies the Hamilton-Jacobi equation[9] $\nabla^\alpha S \nabla_\alpha S = m^2$ (so that $T^\alpha = m^{-1}\nabla^\alpha S$ is tangent to a timelike geodetic congruence) and ρ is a slowly varying amplitude satisfying

$$T^{\alpha}\nabla_{\alpha}\rho+\tfrac{1}{2}\rho\nabla_{\alpha}T^{\alpha}=0. \quad (2)$$

If T^{α} is future (past) pointing then φ will be called a *positive- (negative-) frequency* JWKB wave.

A general (classical) solution ψ of Eq. (1) can be decomposed (approximately) into its positive and negative frequency parts in V by writing

$$\psi=\psi^{+}+\psi^{-}+O(\hbar), \quad (3)$$

where ψ^{+} is a linear superposition of positive-frequency JWKB waves of the form

$$\psi^{+}=\int_k \chi(k)\rho_k \exp(iS_k/\hbar)\,d^3k, \quad (4)$$

and $\psi^{-}=\bar{\psi}^{+}$. Here S_k, $k\in R^3$, is some fixed three-parameter family of solutions of the Hamilton-Jacobi equation (with each $\nabla^{\alpha}S_k$ future pointing) and ρ_k is some corresponding family of (suitably normalized[10]) amplitudes satisfying Eq. (2) [expressions like Eq. (4) can be manipulated in much the same way as Fourier integrals, using the method of stationary phase: see Hörmander[11] and Duistermaat[12]]. A different choice of S_k will result in a different decomposition, but it follows from the principle of stationary phase that the two decompositions will agree in the zeroth order in $\hbar$. Thus one can use Eqs. (3) and (4) to set up a Fock space for Ψ, and to define an (approximate) vacuum state for the system in V.

Now suppose that M contains two Cauchy hypersurfaces σ_1 and σ_2 (with σ_1 to the past of σ_2) near which the radius of curvature is very much greater than $\hbar/m$ while between σ_1 and σ_2 there is some finite region W in which there is a highly oscillating gravitational field.

Near σ_1 and σ_2, the JWKB wave functions $\varphi_k=\rho_k\exp(iS_k/\hbar)$ satisfy Eq. (1) (up to terms of order $\hbar^2$), but in the interaction region W this approximation is not valid. Thus, if one decomposes an *exact* classical solution of Eq. (1) in the form (3) near σ_1 and near σ_2 the results will be different and the two vacuum states, $|0_1\rangle$ near σ_1 and $|0_2\rangle$ near σ_2, will be related by a nontrivial Bogoliubov transformation.[2,13] If the transformation coefficients are small[14] then (up to phase)

$$|0_2\rangle=|0_1\rangle+\int_{k,k'}\bar{\beta}_{kk'}a_k^{-}a_{k'}^{-}|0_1\rangle d^3k\,d^3k', \quad (5)$$

where $\bar{\beta}_{kk'}=(i/2)\Omega(\bar{\psi}_k,\bar{\psi}_{k'})$ and a_k^{-} and $a_{k'}^{-}$ are the creation operators for φ_k and $\varphi_{k'}$. Here ψ_k and $\psi_{k'}$ are *exact* classical solutions of Eq. (1) with $\psi_k=\varphi_k$ near σ_1 and $\psi_{k'}=\varphi_{k'}$ near σ_2 and Ω is the standard symplectic product for scalar fields.[15]

A short calculation gives

$$\beta_{kk'}=(4i\hbar)^{-1}\int_W[(\rho_k\Box\rho_{k'}+\rho_{k'}\Box\rho_k)\,e^{2iS/\hbar}]\,d\omega, \quad (6)$$

where $d\omega$ is the metric volume element and $2S=S_k+S_{k'}$. This can be interpreted as (half) the amplitude for creating a pair of particles with wave functions φ_k and $\varphi_{k'}$ near σ_2.

The physical content of Eq. (6) is illustrated by a computation of the pair production in the threshold limit, that is when $\nabla_{\alpha}(S_k-S_{k'})$ is small compared with $\nabla_{\alpha}S$, so that most of the energy of the pair is in the two rest masses. It follows from the equation of geodesic deviation[16] that, for each k,

$$D_k\theta_k=-2\sigma_k^{\,2}-\theta_k^{\,2}/3-R_{\alpha\beta}T_k^{\,\alpha}T_k^{\,\beta}, \quad (7)$$

where $T_k^{\,\alpha}=m^{-1}\nabla^{\alpha}S_k$, $D_k=T_k^{\,\alpha}\nabla_{\alpha}$, $\theta_k=\nabla_{\alpha}T_k^{\,\alpha}$, and σ_k is the shear of the $T_k^{\,\alpha}$ congruence. With this, the highest-order contribution to $\beta_{kk'}$ is found to be

$$\beta_{kk'}=(4i\hbar)^{-1}\int_W[\rho_k\rho_{k'}R_{\alpha\beta}T^{\alpha}T^{\beta}\exp(2iS/\hbar)]\,d\omega, \quad (8)$$

where $T^{\alpha}=m^{-1}\nabla^{\alpha}S$. This has two promising features. First, the dominant term involves only the Ricci curvature: As in electrodynamics, to the first order of approximation, it is the source of the external field which creates particles. Pure gravitational radiation (with $R_{\alpha\beta}=0$) has a much smaller effect, contributing to the oscillations in $\Box\rho_k$ and $\Box\rho_{k'}$ only by first shearing the two geodetic congruences. Secondly, to have any particle production at all (in this order of approximation) the function $R_{\alpha\beta}T^{\alpha}T^{\beta}$ must contain components which oscillate like $\exp(-2iS/\hbar)$; that is, the gravitational field in W must contain frequencies greater than $2m/\hbar$ (again, as in electrodynamics).

A number of technical matters will be dealt with in more detail elsewhere using geometric quantization theory[17]—in particular, the global behavior of the JWKB wave functions, the precise meaning of Eq. (4) (it is, in fact, a BKS transform[17,18] from the wave function space of the polarization defined locally by S_k), and the extension to particles with spin (the geodetic congruences are then replaced by congruences of Papapetrou trajectories[19]).

I thank Arthur Wightman and Larry Smarr for

their interest and encouragement.

*Supported in part by the National Science Foundation Grant No. GP30799X.

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[17]D. J. Simms and N. M. J. Woodhouse, "Lectures on Geometric Quantization," Springer Lecture Notes in Physics (to be published), and references therein.
[18]R. Blattner, in *Quantization and Representation Theory*, AMS Proceedings of Symposia in Pure Mathematics, Williamstown, Massachusetts, 1972 (American Mathematical Society, Providence, R. I., 1973), Vol. 26, p. 147. BKS stands for Blattner, Kostant, and Sternberg.
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Approach to Scaling of the Structure Functions of Neutrons and Protons*

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Differences in the approach to scaling of neutron and proton structure functions should result from the mass splitting of SU(6) multiplets. I verify the existence of this effect and use it to extract from recent data the form of the scaling structure functions. These functions are found to vanish at $x=1$ as $(1-x)^3$ with a ratio $F_n(x)/F_p(x)$ of $\frac{1}{4}$.

Deep inelastic scattering experiments[1] provide information on the null-plane[2] (or infinite-momentum) wave function of the nucleon. If the impulse approximation of the parton model[3] were exact, this information could be summarized in terms of scaling structure functions[4]

$$F(x_0) = \lim_{q^2\to\infty} F(x_0, q^2), \qquad (1)$$

where x_0 is the usual Feynman scaling variable

$$x_0 = \omega_0^{-1} = -q^2/2m_N\nu. \qquad (2)$$

To extract scaling structure functions from data at moderate q^2 it is necessary to parametrize the q^2 dependence of $F(x_0,q^2)$. A simple phenomenological form for this dependence was suggested by Bloom and Gilman[5] in terms of a variable x_M given by

$$x_M^{-1} = \omega_M = \omega_0 - M^2/q^2. \qquad (3)$$

At finite q^2 one approximates $F(x_0,q^2)$ in the form $F(x_M)$.

Initial fits using this variable indicated the following general features: (i) Neutron and proton data were consistently fit with $M^2 = m_p^2 = 0.88$ GeV^2. (ii) Both neutron and proton data seemed consistent with the behavior

$$F(x) \sim (1-x)^3$$

for x near 1, as had been predicted on the basis of the parton model by Drell, Yan, and West.[6] (iii) The neutron/proton ratio $F_n(x)/F_p(x)$ seemed[7] to fall toward $\frac{1}{4}$ as x approached 1. This last result seemed surprisingly inconsistent with the SU(6) prediction $F_n(x)/F_p(x) \to \frac{2}{3}$ as $x\to 1$. In a previous paper,[8] however, I have argued that the SU(6) symmetry breaking effects which split the masses of the nucleon and the $\Delta(1232)$ resonance modify this prediction from $\frac{2}{3}$ to $\frac{1}{4}$ and account for