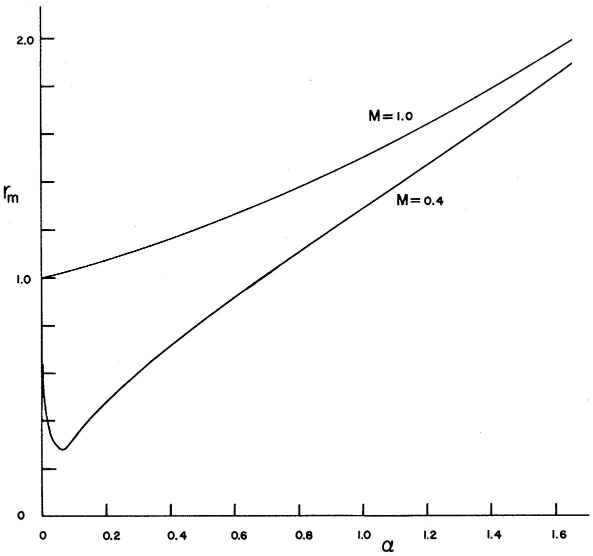

FIG. 2. The maximum field amplitude r_m characterizing the solutions in nondepleted regions versus the ion density $\alpha = \omega_{p0}^2/\omega^2$. The depleted regions lie above the curves.

waves in inhomogeneous plasmas. When $\alpha(\zeta)$ is a slowly varying function of ζ, the following integral is an adiabatic invariant[5]:

$$I_r = \oint P_r dr = 4\int_0^{\zeta_p} [2E - M^2/r^2 - r^2 + 2\alpha(1+r^2)^{1/2}]\, d\zeta, \quad (13)$$

where the integral is taken over the path for given E and α. By setting the adiabatic invariant I_r equal to the adiabatic integral for the incoming wave, $I_{0r} = \oint P_{0r}\, dr_0 = \oint (2E_0 - M^2/r_0^2 - r_0^2)^{1/2} dr_0$, one is able to obtain the dependence of the ion density α on the incident vacuum amplitude. When $M \neq 0$, this situation may occur in plasmas of some sinusoidal ion-density variations. However, the calculation in this case is quite complicated and will be presented elsewhere. When $M = 0$, the estimation of the maximum amplitude r_m presented above reduces to the standing-wave result of Marburger and Tooper,[4] which corresponds to the case of monotonically increasing ion-density variation.

In conclusion, I have presented a class of exact general solutions for strong transverse waves in a cold overdense plasma, which incorporates both the exact traveling-wave solutions and the exact standing-wave solutions. The analytic solutions for the circularly polarized waves are thus suitable for the treatment of inhomogeneous plasmas with any reflection.

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Envelope Solitons in the Presence of Nonisothermal Electrons*

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A set of equations describing the coupling of high-frequency electrostatic waves with ion fluctuations is obtained taking into account a nonisothermal state for the electrons. Stationary envelope solitons are found with narrow widths, slow velocities, and strong field intensities. For their existence the field intensity has to exceed a threshold which is determined by the number of reflected electrons.

The problem of generation of localized electric fields and density cavities has been of much interest[1-6] in connection with the heating of plasmas by a high-power laser or an electron beam. The mechanism by which envelope solitons are produced is the ponderomotive force exerted by the high-frequency waves on the slow ion motion. As a result, there appears a density cavity in which high-frequency waves become trapped. Two types of stationary rarefaction solitons were discussed in this context. The first soliton[2] has a density depression proportional to $\epsilon^2 \propto m_e/m_i$

and moves with a group velocity v_g less than the sound speed $c_s=(T_e/m_i)^{1/2}$. The second soliton[3] has a larger density depression ($\delta n/n_0 \propto \epsilon$) and moves with a velocity slightly less than the sound speed. In the latter case where $v_g = c_s[1-O(\epsilon)]$ additional nonlinear terms in the ion equations become important. For both cases the analysis was based on the assumption that the electrons obey an isothermal equation of state. This means that the electrons reflected by the negative effective potential follow an unperturbed Maxwell distribution. On the other hand, trapping vortices when observed in computer simulations are usually represented by flat or concave distributions for the trapped (reflected) electrons. Deviations from the isothermal state have recently been reported by Valeo and Kruer[4] who followed the formation of solitons in a driven plasma. In this Letter we investigate the consequences arising from such type of electron behavior. We will show that a different nonlinearity in the ion equations arises, leading to a new envelope soliton solution.

The basic equations are derived in the usual manner.[6] Averaging the electron equation of motion over the high-frequency oscillations, we obtain

$$\frac{d|\tilde{v}_e|^2}{dx} = \frac{e}{m_e}\frac{d\Phi_s}{dx} - \frac{1}{m_e n_s}\frac{dp_s}{dx}, \tag{1}$$

where the subscript s denotes the slowly varying part of the corresponding quantities, and $\tilde{v}_e$ is introduced by $v_e = \tilde{v}_e \exp(-i\omega_{pe}t) + \text{c.c.}$, the high-frequency-induced electron velocity. We introduce an effective potential $\varphi \equiv \varphi_s - \varphi_p$ which is normalized by the electron thermal energy in the unperturbed state ($\varphi_s = e\Phi_s/T_e$). Here $\varphi_p = m_e|\tilde{v}_e|^2/T_e$ is the ponderomotive potential. Equation (1) can be satisfied by assuming an electron equation of state, namely $p_s = p_s(\varphi)$, where $dp_s/d\varphi = p_0 n_s(\varphi)/n_0$, and $p_0 = n_0 T_e$. Thus if $n_s(\varphi)$ is known $p_s(\varphi)$ follows by integration. To get $n_s(\varphi)$, we refer to an appropriate Bernstein-Greene-Kruskal (BGK) solution because the slowly varying perturbation can be considered to be stationary with respect to the electrons. A BGK solution is considered to be adequate if it represents the nonlinear version of the natural eigenmodes of the plasma, e.g., in our case the ion-acoustic oscillations (rather than van Kampen modes). As has been shown earlier,[7] this means that the electron distribution function has to satisfy some regularity requirements.[8] These are fulfilled if one assumes for both trapped and untrapped electrons Maxwellian distributions which are joined continuously at the boundary of the trapped electrons. To describe also flat-topped distributions or distributions with a dip at the center, we allow for a different temperature T_{et} for the trapped-electron distribution ($\beta = T_e/T_{et}$). The normalization is chosen such that n_s becomes n_0 in the zero-field region ($x=\pm\infty$). Here we are interested in a situation where $\varphi(x)$ represents a negative potential dip. Thus some electrons coming from infinity moving slowly in the comoving frame will be reflected from this region. In our terminology these are the trapped electrons. We obtain[7] by integrating the distribution function

$$n_s(\varphi) = Bn_0\{I(\psi+\varphi) + |\beta|^{-1/2}\times\begin{cases}\exp[\beta(\psi+\varphi)]\,\text{erf}([\beta(\psi+\varphi)]^{1/2})\}, & \beta\geq 0,\\ 2\pi^{-1/2}W([-\beta(\psi+\varphi)]^{1/2})\}, & \beta<0,\end{cases} \tag{2}$$

where B is the quantity inside the curly braces at $\varphi=0$, and $-\psi$ is the depth of the potential. The function $I(x)$ is defined by $I(x)=\exp(x)[1-\text{erf}(x^{1/2})]$ and $W(x)$ is the Dawson integral, namely $W(x)=\exp(-x^2)\times\int_0^x dt\,\exp(t^2)$. If we set $\beta=1$, we have $n_s = n_0\exp\varphi$, the usual isothermal result. We are interested in more flat distributions for which β is nearly zero. For small amplitudes $\psi \ll 1$, Taylor expansion of (2) gives

$$n_s(\varphi) = n_0\{1+\varphi+\tfrac{4}{3}b[\psi^{3/2}-(\psi+\varphi)^{3/2}]\} + O(\psi^2), \tag{3}$$

where $b=(1-\beta)\pi^{-1/2}$. From (3) it follows that for a given $\varphi(x)$ the density depression is less pronounced than in the isothermal case (or in other words, there are fewer particles being expelled from the high-frequency field intensity region). We relate φ with the ion-density fluctuation δn by means of the slowly varying part of Poisson's equation

$$\lambda_e^2\frac{\partial^2\varphi_s}{\partial x^2} = \frac{n_s}{n_0} - \left(1+\frac{\delta n}{n_0}\right), \tag{4}$$

where λ_e is the electron Debye length. Inserting (3) into (4) and retaining terms up to second order we find

$$\varphi_s=\left(1+\lambda_e^2\frac{\partial^2}{\partial x^2}\right)\frac{\delta n}{n_0}+\varphi_p-\frac{4b}{3}\left[\left(-\frac{\delta n_M}{n_0}\right)^{3/2}-\left(-\frac{\delta n_M}{n_0}+\frac{\delta n}{n_0}\right)^{3/2}\right]. \qquad (5)$$

In the above, the dispersive term is thereby assumed to be small and δn_M represents the maximum depth of the density trough. Finally, substituting (5) into the linearized equations for the ions, we obtain

$$\left[\frac{\partial^2}{\partial t^2}-c_s^2\frac{\partial^2}{\partial x^2}\left(1+\lambda_e^2\frac{\partial^2}{\partial x^2}\right)\right]\frac{\delta n}{n_0}=c_s^2\frac{\partial^2}{\partial x^2}\left\{\frac{|E|^2}{4\pi n_0T_e}-\frac{4b}{3}\left[\left(-\frac{\delta n}{n_0}\right)^{3/2}-\left(-\frac{\delta n_M}{n_0}+\frac{\delta n}{n_0}\right)^{3/2}\right]\right\}. \qquad (6)$$

In (6) we have substituted $|\tilde{v}_e|=|eE|/m_e\omega_{pe}$, where E is the slowly varying complex amplitude of the high-frequency wave. For the electrostatic waves, the latter is governed by[6]

$$\left(i\frac{\partial}{\partial t}+\frac{3}{2}\omega_{pe}\lambda_e^2\frac{\partial^2}{\partial x^2}\right)E=\frac{\omega_{pe}}{2}\left(\frac{\delta n}{n_0}\right)E, \qquad (7)$$

where ω_{pe} is the electron plasma frequency. We observe that the strength of the ponderomotive force is weakened by the appearance of the new nonlinear term. Thus, for waves propagating near the sound speed when the nonlinear term becomes important, a larger field intensity is required to produce the same density cavity. To study these phenomena in detail, let us introduce the following scaling:

$$\delta n/n_0=-\epsilon\nu(\xi,\tau), \qquad (8a)$$

$$\frac{E}{(4\pi n_0T_e)^{1/2}}=\epsilon^{3/4}W(\xi,\tau)\exp[i\theta(\tau')], \qquad (8b)$$

where for smallness parameter ϵ, which is a measure of the ion-density depression $(0\le\nu\le1)$, we set $\epsilon=\eta(m_e/m_i)^{1/2}$ and $\eta\gtrsim1$. The independent variables in (8) are $\xi=\epsilon^{1/2}\lambda_e^{-1}(x-c_st)$, $\tau=\epsilon\tau'$ $=\epsilon\omega_{pi}t$, where ω_{pi} is the ion-plasma frequency. Then, to lowest order in ϵ, we get from (6) and (7)

$$\frac{\partial\nu}{\partial\tau}=\frac{1}{2}\frac{\partial W^2}{\partial\xi}+\frac{2b}{3}\frac{\partial(1-\nu)^{3/2}}{\partial\xi}, \qquad (9)$$

$$-W\eta^{-1}\frac{\partial\theta}{\partial\tau'}+\frac{3}{2}\frac{\partial^2W}{\partial\xi^2}=-\frac{\nu W}{2}. \qquad (10)$$

We note that in (9), which in the isothermal case is a Korteweg–de Vries type equation, the ion dispersion term disappeared because of our new scaling. Only the group dispersion of the high-frequency waves will play a role.

We now look for a stationary soliton solution. We introduce a new dependent variable $v=1-\nu$ and a nonlinear frequency shift $A=2\eta^{-1}\partial\theta/\partial\tau'$, and assume $v=v(\xi-\alpha\tau)$. Then from (9) and (10) we obtain

$$\alpha\frac{\partial v}{\partial\xi}=\frac{1}{2}\frac{\partial W^2}{\partial\xi}+\frac{2b}{3}\frac{\partial v^{3/2}}{\partial\xi}, \qquad (11)$$

$$\partial^2W/\partial\xi^2=(A-1+v)W/3. \qquad (12)$$

Integration of (11) with $W=W_0$ at $v=0$ (corresponding to the maximum electric field intensity of the envelope soliton) and $W=0$ at $v=1$ $(x=\pm\infty)$ yields

$$W^2=W_0^2(1-v^{1/2})(1+v^{1/2}+v/\rho), \qquad (13)$$

$$\alpha=-W_0^2(1-\rho^{-1})/2,\quad \rho=3W_0^2/4b. \qquad (14)$$

Equation (14) shows that in the laboratory frame the envelope soliton will move with a subsonic velocity

$$v_g=c_s[1-0.5\epsilon^{1/2}W_0^2(1-\rho^{-1})]$$

which is less than the soliton velocity for isothermal electrons.

From (11) we have

$$WW_\xi=v_\xi(\alpha-bv^{1/2}), \qquad (15)$$

where $W_\xi\equiv\partial W/\partial\xi$ and $v_\xi\equiv\partial v/\partial\xi$. Integrating (12) and using (15) we obtain

$$\tfrac{1}{2}W_\xi^2=\tfrac{1}{6}W^2(A-1)+\tfrac{1}{6}v^2(\alpha-4bv^{1/2}/5)+C. \qquad (16)$$

Since $W_\xi=0$ at $v=0$ we find $C=W_0^2(1-A)/6$, and $W_\xi=0$ at $v=1$ $(W=0)$ determines the frequency shift A in terms of ρ, i.e., $A=\frac{1}{2}[1-(5\rho)^{-1}]$. Equations (13) and (16) combine to yield

$$W_\xi^2=v(1-v^{1/2})Q(v), \qquad (17)$$

where

$$Q(v)=W_0^2\rho[(\rho+0.2)(1-\rho^{-1}+v^{1/2})+1.2v]/6.$$

From (13), (15), and (17), after some algebra, one obtains

$$v_\xi^2/2=W^2W_\xi^2/2(\alpha-bv^{1/2})^2\equiv-V(v). \qquad (18)$$

The classical potential $V(v)$ is given by

$$V(v)=-v(1-v^{1/2})^2G(v,\rho), \tag{19}$$

where

$$G(v,\rho)=\frac{[v+\rho(1+v^{1/2})]\{(\rho+0.2)[v^{1/2}+(\rho-1)/\rho]+1.2v\}}{3(1.5v^{1/2}+\rho-1)^2}. \tag{20}$$

From (19) it follows immediately that a rarefaction soliton for v (or $\delta n/n_0$) exists provided $G(v,\rho)$ is positive definite. Since $G(0,\rho)=(\rho+0.2)/3(\rho-1)$, we get the existence condition $\rho>1$. Hence an envelope soliton exists if the trapping parameter b is smaller than $3W_0^2/4$. We conclude that the stronger the high-frequency field intensity, the larger the deviation from isothermality can be.

The general solution of (18) for arbitrary ρ is rather complicated. Two particular solutions are easily obtained. By choosing $\rho=1.5$, the function $G(v,\rho)$ becomes approximately constant; i.e., $G(v,\rho=1.5)=8k_0^2\approx0.83$. Thus (18) can be integrated to yield

$$v(\xi)=4\sinh^2(k_0|\xi|)\exp(-2k_0|\xi|). \tag{21}$$

Transforming back to the original variables one obtains

$$\delta n/n_0=-\epsilon[1-4\sinh^2(k_0|\xi|)\exp(-2k_0|\xi|)]. \tag{22}$$

The high-frequency electric field of the envelope soliton is given by

$$W(\xi)=W_0\exp(-k_0|\xi|)\{2-\exp(-2k_0|\xi|)[1-(4/\rho)\sinh^2(k_0|\xi|)]\}^{1/2}. \tag{23}$$

Next consider the isothermal limit [$b\to0$ or $\rho\to\infty$ in (19)]. Accordingly, we have $G(v,\infty)=(1+v^{1/2})^2/3$, and $V(v)=-v(1-v)^2/3$. The corresponding soliton is given by

$$\delta n/n_0=-\epsilon\ \mathrm{sech}^2 y;\quad E/(4\pi n_0T_e)^{1/2}=\epsilon^{3/4}W_0\,\mathrm{sech}\,y\exp[i\epsilon\omega_{pe}t/4], \tag{24}$$

where

$$y=(\epsilon/6)^{1/2}[x/\lambda_e-(1-\epsilon^{1/2}W_0^2/2)\omega_{pi}t].$$

Thus our solution with three independent parameters (ϵ, W_0, ρ) reduces in the isothermal limit to that obtained by Karpman.[9] The one-parametric soliton solution of Nishikawa *et al.*[3] is completely different. The electric field has a node at density minimum instead of a maximum. It is obtained by a different scaling [i.e., $|E|^2/4\pi n_0T_e=O(\epsilon^2), \tau=\epsilon^{3/2}\omega_{pi}t$], for which the dispersion term as well as the bilinear term $(\delta n/n_0)^2$ must be retained in (6). We note that our results are valid in the parameter range $(m_e/m_i)^{1/2}\ll\epsilon^{1/2}\ll1$. The first inequality originates from the neglect of the imaginary part of (7) (i.e., $\epsilon^{1/2}/\eta\ll1$), whereas the second one is due to the neglect of the bilinear terms. Let us finally compare our results with the standing spikes observed in the experiments of Wong and Quon.[5] The experimental values (i.e., $|E|^2/4\pi n_0T_e=0.2$, $\delta n/n_0=0.1$) can be fitted by choosing $\eta=25$ (i.e., $\epsilon=0.1$ for an argon plasma), and $W_0=2.5$. For these values the speed is found to be $v_g=c_s/\rho$ which vanishes in the isothermal limit. Also the soliton width of $8\lambda_e$ fits well with the experimental value of $10\lambda_e$. However, we mention that the assumption $\epsilon^{1/2}\ll1$ is not rigorously satisfied, and one should extend the isothermal soliton into the finite-amplitude regime. It is found[10] that the observations of Wong and Quon[5] as well as the more recent experiments of Ikezi, Chang, and Stern[11] can be explained.

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Stability of Field-Reversed Ion Rings in a Background Plasma*

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The low-frequency stability of a field-reversed ion ring/layer of large Larmor radius in a background plasma is treated by a generalized form of the "energy principle."

Recent advances in the production of multikiloampere ion beams[1] are applicable to the creation of field-reversed ion rings/layers by single pulse injection in the manner already demonstrated for E layers.[2] It has been pointed out[3] that the field-reversal factor $\zeta \equiv \delta B/B_0 = Nr_iR_L/R^2$ can be increased by adiabatic magnetic compression, i.e., $\zeta \propto B_0^{1/2}$; here B_0 is the external field, δB is the diamagnetic change in the field on axis, N is the total number of ring ions, $r_i = e^2/mc^2$ is the ion classical radius, R is the mean ring radius, and $R_L = mV_0/eB_0$ is the ion Larmor radius in the external field B_0. Thus injection of a pulse of intense ion beam followed by magnetic compression may be a reasonable technique for the production of a field-reversed ion ring. The low-frequency stability of the combined ion-ring/background-plasma system has not yet been satisfactorily resolved although there have been studies of (i) specific modes,[4] with and without a background plasma, and (ii) stability of plasma confined by ion ring/E layer but with the ring/layer assumed to be rigid.[5] More recently Lovelace[6] has addressed himself to the stability of the combined system. The present calculation is more general and differs from his in that the Vlasov formalism is employed for the ion-ring dynamics as opposed to his approach which limits consideration to rigid displacements of the center of mass of the ring cross section.

We resolve the system stability by a technique employed initially by Newcomb[7] and applied subsequently by Davidson and Krall[8] to unneutralized E layers. The addition of a background plasma enormously complicates the problem. We give below the main results of the analysis, deferring the details to a future publication. Since our objective is to obtain an energy principle that furnishes sufficient conditions for stability we begin by recognizing the following constant of motion of system, viz.,

$$C = U - (\Omega L_z - \int d^3x\, d^3v f \ln f),$$

where the total system energy U is composed of plasma kinetic energy $\int d^3x\, \frac{1}{2}nmu^2$, pressure energy $\int d^3x\, p/(\gamma - 1)$, electromagnetic energy $\int d^3x \times (|\vec{E}|^2 + |\vec{B}|^2)/8\pi$ and the ring kinetic energy $\int d^3x \times d^3v\, \frac{1}{2}mv^2 f$, where $f(\vec{x}, \vec{v}, t)$ is the ring distribution; L_z is the z component of the total angular momentum,

$$L_z = \int d^3x\, \hat{z}\cdot\{nm\vec{r}\times\vec{u} + (4\pi c)^{-1}\vec{r}\times(\vec{E}\times\vec{B}) + m\int d^3v\, \vec{r}\times\vec{v}f\},$$

and Ω is an arbitrary constant. The quantity $\Omega L_z - \int d^3x\, d^3v f \ln f$ may be regarded as a generalized entropy of the ion ring in which case C is interpreted to be the system free energy when the plasma pressure is neglected. In what follows we limit our discussion to a pressureless cold plasma governed by the continuity equations, the plasma ion momentum balance,

$$nm\, d\vec{u}/dt = ne(\vec{E} + \vec{u}\times\vec{B}), \qquad (1)$$