Strong Transverse Electromagnetic Waves in Overdense Plasmas*

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I present a new class of exact solutions for strong transverse electromagnetic waves in a cold overdense plasma, which incorporates both the exact traveling-wave and standing-wave solutions. The waves are circularly polarized, and the dielectric constant of the plasma is a function of z.

Analytical studies of strong electromagnetic waves propagating in a cold overdense plasma are generally based on Maxwell's equations and the relativistic equation of motion for electrons.[1] In the absence of the reflected wave the exact traveling-wave solutions of Akhiezer and Polovin are found from these basic equations.[1-3] Recently, Marburger and Tooper[4] have obtained another exact standing-wave solution for both homogeneous and inhomogeneous cold plasmas, which are suitable for the investigation of the extreme case of total reflection.

In this Letter I present a class of exact general solutions for strong transverse electromagnetic waves propagating in both homogeneous and inhomogeneous cold plasmas. The solutions can be written in a closed form for the homogeneous case. The waves corresponding to these solutions are circularly polarized, and the dielectric constant of the plasma is generally a function of z. If the reflection is negligible, the solutions reduce to the usual circularly polarized traveling-wave solutions.[1,2] The exact standing-wave solutions of Marburger and Tooper[4] are also recoverable from the solutions in the case of total reflection.

The basic equations for strong electromagnetic waves propagating in a cold overdense plasma are Maxwell's equations for the electromagnetic field and the relativistic electron equation for fixed ions:

$$\nabla\times\vec{B}=\frac{4\pi}{c}ne\vec{v}+\frac{1}{c}\frac{\partial\vec{E}}{\partial t},\quad \nabla\cdot\vec{B}=0,$$
$$\nabla\times\vec{E}=-\frac{1}{c}\frac{\partial\vec{B}}{\partial t},\quad \nabla\cdot\vec{E}=4\pi e(n-n_0),\tag{1}$$
$$\frac{\partial\vec{p}}{\partial t}+(\vec{v}\cdot\nabla)\vec{p}=e\vec{E}+\frac{e}{c}\vec{v}\times\vec{B},$$

where $\vec{p}=m\vec{v}(1-v^2/c^2)^{1/2}$ is the relativistic electron momentum, $n_0(z)$ is the ion density, and the independent variables are z and t. In the case of a standing wave, it has been shown from Eqs. (1) that $p_z=0$ and the wave is circularly polarized.[4] The pure transverse traveling wave existing in the homogeneous plasma is also circularly polarized.

Let us look for general transverse-wave solutions of Eqs. (1) which incorporate both the pure transverse traveling wave and the standing wave. Here we assume that the transverse components of $\vec{E}$ and $\vec{B}$ can be derived from a vector potential $\vec{A}$ of the form

$$\frac{e\vec{A}}{mc^2}=r(\zeta)\{-\hat{x}\sin[\omega t-\varphi(\zeta)]+\hat{y}\cos[\omega t-\varphi(\zeta)]\},\tag{2}$$

where $\zeta=\omega z/c$. The longitudinal component of $\vec{E}$, however, can be obtained from a time-independent scalar potential. Inserting the relevant expressions into Eqs. (1) we find

$$\alpha_e-\alpha=d^2(1+r^2)^{1/2}/d\zeta^2,\tag{3}$$
$$\ddot{r}-r\dot{\varphi}^2+r^2=\alpha_e r/(1+r^2)^{1/2},\tag{4}$$
$$r\ddot{\varphi}+2\dot{r}\dot{\varphi}=0,\tag{5}$$

where $\dot{X}$ denotes $dX/d\zeta$, etc., and α_e and α are defined by

$$\alpha_e=\frac{\omega_p^2}{\omega^2}=\frac{4\pi ne^2}{m\omega^2},\quad \alpha=\frac{\omega_{p0}^2}{\omega^2}=\frac{4\pi n_0e^2}{m\omega^2}.\tag{6}$$

From Eq. (5) it is easy to show that $M=r^2\dot{\varphi}$ is a constant of motion with respect to the variation in ζ, and φ varies monotonically with ζ. The dielectric constant ϵ of the plasma can also be related to $\dot{\varphi}$ by the relation $\epsilon=\dot{\varphi}^2$.

Eliminating α_e from Eqs. (3) and (4), one obtains an equation governing the amplitude r. Here we notice that Eqs. (4) and (5) are derivable from the Lagrangian

$$L=\tfrac{1}{2}\dot{r}^2/(1+r^2)+\tfrac{1}{2}r^2\dot{\varphi}^2-\tfrac{1}{2}r^2+\alpha(1+r^2)^{1/2},\tag{7}$$

where, as before, the dot denotes the differentiation with respect to the "time" variable ζ, and α may be a function of ζ in the case of inhomogeneous plasmas. The associated "momentum" conjugate to r is then $P_r = \dot{r}/(1+r^2)$, and the associated "energy" E is given by

$$E = \tfrac{1}{2}\dot{r}^2/(1+r^2) + \tfrac{1}{2}M^2/r^2 + \tfrac{1}{2}r^2 - \alpha(1+r^2)^{1/2}. \quad (8)$$

When the ion density is homogeneous, the "energy" E is a constant of motion in ζ. In the case of traveling waves, we get from Eq. (2), $\varphi(\zeta) = kz = (ck/\omega)\zeta$. Since $M = r^2\dot{\varphi}$ is a constant of motion, this particular form of $\varphi(\zeta)$ implies that the amplitude r^2 is constant. On the other hand, if we let $\varphi(\zeta) = 0$ as required by the case of standing waves, we find that $M = 0$ and the Lagrangian given in Eq. (7) reduces to that obtained by Marburger and Tooper.[4] Therefore I conclude that the present form of the Lagrangian governing the equations for the amplitude $r(\zeta)$ and the phase $\varphi(\zeta)$ is more general than those obtained so far for strong transverse electromagnetic waves propagating in a cold overdense plasma.

For the homogeneous ion density, the motion for the amplitude r may be examined by considering a plot of the effective potential $V_{\rm eff}(r) = \tfrac{1}{2}M^2/r^2 + \tfrac{1}{2}r^2 - \alpha(1+r^2)^{1/2}$ against the amplitude r. The condition $\dot{r} = 0$ indicates the turning points of the "path," and the range of r falls between the limits $r_{\rm min}$ and $r_{\rm max}$. The path is not closed in the present case.[5] As $M \neq 0$, the amplitude r can never be zero except for the case of standing waves. By letting $u^2 = 1 + r^2$ we obtain, from Eq. (8), the relation

$$\zeta = du\left[-u^4 + 2\alpha u^3 + 2(E+1)u^2 - 2\alpha u - 2E - M^2 - 1\right]^{-1/2} + \text{const.} \quad (9)$$

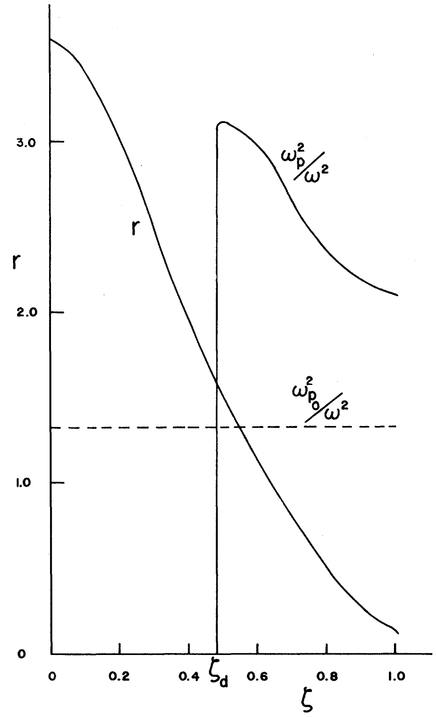

FIG. 1. The transverse electric field r and the electron density $\omega_p{}^2/\omega^2$ versus $\zeta = \omega z/c$ over a quarter period ζ_p with the values of the parameters $E = 1.5$ and $M = 0.3$. The ion density $\omega_{p0}{}^2/\omega^2$ is chosen to be $\frac{4}{3}$.

Thus the amplitude r can be expressed explicitly in terms of Jacobi's elliptic functions. The formulas for r as a function of ζ are more complicated than those obtained for the standing waves and will be given elsewhere. Once $r = r(\zeta)$ is known, the phase $\varphi(\zeta)$, the electron density $\alpha_e(\zeta)$, and the spatial quarter period ζ_p can in turn be calculated. As an example of the solutions, I present, in Fig. 1, the dependences of the amplitude r and the electron density α_e on the spatial variable ζ_p for the value of $\alpha = \frac{4}{3}$.

From Eqs. (3) and (4) it is easy to show that when the following condition is satisfied,

$$M^2(1+r^{-2}) + \dot{r}^2 - r^2(1+r^2) < -\alpha(1+r^2)^{3/2}, \quad (10)$$

the electron density can vanish. This may occur if the values of r are near $r_{\rm max}$ and M is small. Where there are no electrons, the Lagrangian reduces to the form in a vacuum,

$$L_0 = \tfrac{1}{2}\dot{r}_0{}^2 + \tfrac{1}{2}r_0{}^2\dot{\varphi}^2 - \tfrac{1}{2}r_0{}^2. \quad (11)$$

In this case, the associated momentum conjugate to r_0 is $P_{0r} = \dot{r}_0$ and the associated energy is given by $E_0 = \tfrac{1}{2}\dot{r}_0{}^2 + \tfrac{1}{2}M^2/r_0{}^2 + \tfrac{1}{2}r_0{}^2$. The boundary condition at the depletion boundary requires that the logarithmic derivatives of the solutions in two regions match at the boundary. The value r_d of the amplitude at the boundary between the depleted and the nondepleted regions can then be obtained from the equation

$$\frac{1}{G}\frac{(E_0{}^2 - M^2)^{1/2}\sin(2G)}{E_0 + (E_0{}^2 - M^2)^{1/2}\cos(2G)} = \alpha\frac{(1+r_d{}^2)^{1/2}}{r_d{}^2}, \quad (12)$$

where $G^2 = 2(r_d/\alpha)^2[E - V_{\rm eff}(r_d)]$. In Fig. 2, I show the dependences of r_d versus α for the particular case with maximum amplitude $r_m = r_d$. The depletion region lies above the curve.

The solutions for strong transverse waves found in Eq. (2) can be employed in the investigation of the properties of the electromagnetic

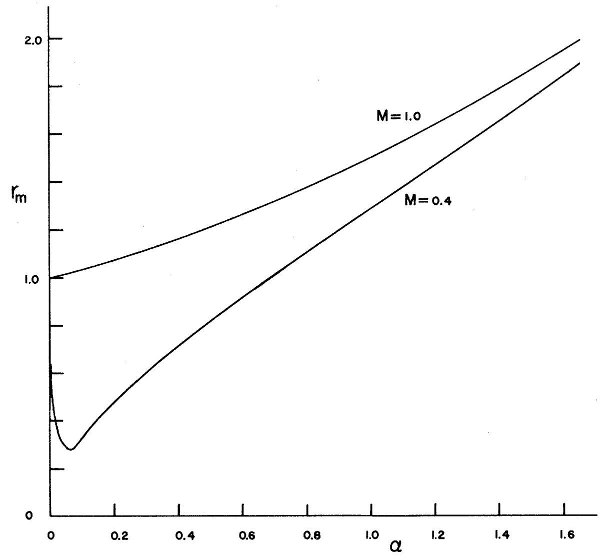

FIG. 2. The maximum field amplitude r_m characterizing the solutions in nondepleted regions versus the ion density $\alpha = \omega_{p0}{}^2/\omega^2$. The depleted regions lie above the curves.

waves in inhomogeneous plasmas. When $\alpha(\zeta)$ is a slowly varying function of ζ, the following integral is an adiabatic invariant[5]:

$$I_r = \oint P_r dr = 4\int_0^{\zeta_p} [2E - M^2/r^2 - r^2 + 2\alpha(1+r^2)^{1/2}]\, d\zeta, \quad (13)$$

where the integral is taken over the path for given E and α. By setting the adiabatic invariant I_r equal to the adiabatic integral for the incoming wave, $I_{0r} = \oint P_{0r}\, dr_0 = \oint (2E_0 - M^2/r_0^2 - r_0^2)^{1/2} dr_0$, one is able to obtain the dependence of the ion density α on the incident vacuum amplitude. When $M \neq 0$, this situation may occur in plasmas of some sinusoidal ion-density variations. However, the calculation in this case is quite complicated and will be presented elsewhere. When $M = 0$, the estimation of the maximum amplitude r_m presented above reduces to the standing-wave result of Marburger and Tooper,[4] which corresponds to the case of monotonically increasing ion-density variation.

In conclusion, I have presented a class of exact general solutions for strong transverse waves in a cold overdense plasma, which incorporates both the exact traveling-wave solutions and the exact standing-wave solutions. The analytic solutions for the circularly polarized waves are thus suitable for the treatment of inhomogeneous plasmas with any reflection.

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Envelope Solitons in the Presence of Nonisothermal Electrons*

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A set of equations describing the coupling of high-frequency electrostatic waves with ion fluctuations is obtained taking into account a nonisothermal state for the electrons. Stationary envelope solitons are found with narrow widths, slow velocities, and strong field intensities. For their existence the field intensity has to exceed a threshold which is determined by the number of reflected electrons.

The problem of generation of localized electric fields and density cavities has been of much interest[1-6] in connection with the heating of plasmas by a high-power laser or an electron beam. The mechanism by which envelope solitons are produced is the ponderomotive force exerted by the high-frequency waves on the slow ion motion. As a result, there appears a density cavity in which high-frequency waves become trapped. Two types of stationary rarefaction solitons were discussed in this context. The first soliton[2] has a density depression proportional to $\epsilon^2 \propto m_e/m_i$