Explosive Instability of Drift-Cone Modes in Mirror Machines*

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We examine the parametric coupling of the negative-energy drift-cone mode near the ion-cyclotron frequency with lower-frequency positive-energy modes through the electronic nonlinearity. This parametric process is shown to be stabilized by azimuthal asymmetry occurring in the early stage of the buildup. The theory is compared with the experimental observations.

The interplay of a coherent state with one large-amplitude wave and a turbulent state, composed of many waves of random phases, and their respective effects on the charged particles are among the fundamental problems of nonlinear plasma physics. Recent experiments of mirror machines[1,2] have reported very interesting phenomena relating to this problem, besides their importance to plasma-confinement study. In the early stage of the experiment, when the plasma is nonaxisymmetric with an $m=2$ flute distortion due to the initial injection from the fans, a coherent flute mode ($k_\parallel \simeq 0$, $k\rho \gg 1$) near the ion-cyclotron frequency, Ω_i, is observed to propagate in the ion-diamagnetic-drift direction, exhibiting the characteristics of the drift-cone mode. This coherent ion-cyclotron mode does not cause anomalous ion loss, possibly because of superadiabaticity,[3] and the confinement time is determined essentially by Coulomb collisions. At later times, when the plasma reaches an azimuthal symmetry due to the ∇B drift in the minimum-B configuration, the observed spectrum broadens and becomes turbulent with emergence of modes below the ion-cyclotron frequency as well as cyclotron harmonics. The peak of the spectrum shifts to lower frequency and the overall intensity of the fluctuation increases. With this development of turbulence, ion loss is enhanced indicating anomalous velocity-space diffusion. These observations raise the question of the mechanism responsible for the transition from the coherent, single-wave state to the turbulent state of many waves with the intensification of the overall fluctuation level and its quenching by the azimuthal asymmetry. We examine here the explosive parametric instability of the negative-energy drift-cone mode near the ion-cyclotron frequency into low-frequency, positive-energy waves as a possible explanation of these phenomena. This parametric process is then shown to be effectively stabilized by azimuthal asymmetry, consistent with the experimental observation.

Because of the excess free energy of the ion-loss-cone distribution, the plasma in a mirror machine supports both positive- and negative-energy waves. The nonlinear three-wave interaction involving both positive- and negative-energy waves leads to explosive growth of all three. Explosive instability in a mirror machine was first discussed by Rosenbluth, Coppi, and Sudan[4] and the matrix element of coupling due to *ion nonlinearity* was calculated. The level of fluctuation observed in recent experiments, however, does not exceed the electron temperature which is typically 10^{-2} smaller than the ion temperature, i.e., $e\varphi \lesssim T_e \approx 10^{-2} T_i$. The electron nonlinearity thus dominates the ion nonlinearity and determines the growth rate. In the following, we calculate the growth rate and threshold of this explosive parametric instability of the drift-cone modes using the electron nonlinearity. The growth rate is of the order $0.1\Omega_i$ for typical experimental parameters.

Consider an ion-loss-cone distribution

$$f_0 = \frac{[1+(x+v_y/\Omega_i)L^{-1}]}{\pi^{3/2} a_\perp^2 a_\parallel} \left(\frac{V_\perp}{a_\perp}\right)^2 \exp\left[-\frac{V_\perp^2}{a_\perp^2} - \frac{V_\parallel^2}{a_\parallel^2}\right],$$

where $L = (d\ln n/dx)^{-1}$ is the radial density scale length. The dispersion relation of an electrostatic perturbation of the usual WKB form $\tilde{\varphi}(\vec{x}, t)$

$=\varphi\exp(ik_y y+ik_x x-i\omega t)$, in the limit $k\rho_i\gg 1$, is[5]

$$\epsilon(\vec{k},\omega)=1+\frac{\omega_{pe}^2}{\Omega_e^2}-\frac{1}{k^2\lambda_{Di}^2}\left[\frac{\omega_{*i}}{\omega}-(\omega+\omega_{*i})\beta\sum_{l=-\infty}^{\infty}\frac{1}{\omega-l\Omega_i}\right]=0, \tag{1}$$

where $\omega_{*i}=k_y cT_i/eBL$ is the ion-diamagnetic-drift frequency, $\rho_i=a_\perp/\sqrt{2}\Omega_i$, $\lambda_{Di}^2=T_i/4\pi ne^2$, $\Omega_e=eB/m_e c$, $\beta=[2(2\pi)^{1/2}k\rho_i]^{-1}\ll 1$. Note that the sum over l can be written as $\pi\cot(\pi\omega/\Omega_i)$. Equation (1) has been extensively analyzed.[5,6] It is shown that an unstable mode exists for a density gradient exceeding a certain critical value, i.e., $(\rho_i/L)>0.3[m/M+(\Omega_i/\omega_{pi})^2]^{2/3}$. When this critical density gradient is greatly exceeded, as is the case of present experiments, unstable modes with frequency ω and growth rate γ much above Ω_i form a continuum: $\omega=\gamma=(2/\pi)^{1/4}k\rho_i(\rho_i/L)^{1/2}\Omega_i$, for $k\rho_i(\rho_i/L)^{1/2}\gg 1$. That this high-frequency continuum has not been observed is attributed to the turbulent electron viscosity[6] and will be discussed in a separate paper.[7] We examine here only the unstable mode near the ion-cyclotron frequency and stable modes below Ω_i. Keeping only the $l=1$ term in Eq. (1), we find that the ion-cyclotron mode with

$$k_y\rho_i^2/L=k^2\lambda_{Di}^2(1+\omega_{pe}^2/\Omega_e^2)+1/2(2\pi)^{1/2}k\rho_i$$

is unstable with a growth rate $\Omega_i[k^2\lambda_{Di}^2(1+\omega_{pe}^2/\Omega_e^2)/\beta+1]^{-1/2}\ll\Omega_i$ for $k\rho_i\gg(M/m)^{1/3}(1+\Omega_e^2/\omega_{pe}^2)^{1/3}$. This mode has negative energy as $(\omega\partial\epsilon/\partial\omega)_{\omega=\Omega_i}<0$ near resonance. The parametric decay of this ion-cyclotron wave with ω_0, $\vec{k}_0$ into lower-frequency modes is kinematically allowed if the frequency and wave-number matching (resonance) conditions $\omega_0=\omega_1+\omega_2$ and $\vec{k}_0=\vec{k}_1+\vec{k}_2$ are satisfied, where ω_1, ω_2 and $\vec{k}_1$, $\vec{k}_2$ are the frequencies and wave numbers of the lower-frequency modes. The dispersion relation of the stable, lower-frequency modes can be obtained from Eq. (1). Analytically, there is a wave with $\omega_1=\Omega_i/2$ and $k_y/k^2L=(m/2M)(1+\Omega_e^2/\omega_{pe}^2)$, propagating in the ion-diamagnetic-drift direction. Evaluating the Von Laue factor (the ratio of the wave energy to the electrostatic energy) we find

$$\omega\frac{\partial\,\mathrm{Re}\,\epsilon}{\partial\omega}\bigg|_{\omega=\Omega/2}=\frac{1}{k^2\lambda_{Di}^2}\left[\frac{\omega_{*i}}{\Omega_i}\left(2-\frac{\beta\pi^2}{2}\right)-\frac{\pi^2}{4}\beta\right]. \tag{2}$$

Expressing k_y in terms of k, we find that wave energy is negative for

$$(k\rho_i)^3<[\pi^2/8(2\pi)^{1/2}](M_i/m)(1+\Omega_{pe}^2/\omega_{pe}^2)^{-1}.$$

For $k=k_y$ ($k_x=0$), the wave has negative energy if $\rho_i/L<(m_e/M+\Omega_i^2/\omega_{pi}^2)^{3/4}$.

Additionally, there is a modified drift wave (modified by the ion loss cone) with

$$\omega_2=\omega_{*i}/[\beta+k^2\lambda_{Di}^2(1+\omega_{pe}^2/\Omega_{ce}^2)]\ll\Omega_i.$$

The Von Laue factor is $(\omega\partial\,\mathrm{Re}\,\epsilon/\partial\omega)_{\omega_2}=1+\omega_{pe}^2/\Omega_e^2+\beta$, and it is therefore a positive-energy wave.

Because the experimentally observed ion-cyclotron wave has frequency somewhat below Ω_i (typically ω_0 varies from $0.6\Omega_i$ to $0.9\Omega_i$) and $k_0\rho_i^2/L$ is of the order unity with $\vec{k}_0$ along the ion-diamagnetic-drift direction (denoted $\hat{y}$ here), we consider the decay of a negative-energy pump wave with frequency $\omega_0=\Omega_i(1-\delta)$, $1>\delta>\frac{1}{2}$, into a half-harmonic mode (labeled 1) and a low-frequency mode (labeled 2). The three-wave resonance conditions then yield the following relation between the wave vectors k_1 and k_2:

$$\frac{k_0\rho_i^2}{L}=\frac{k_1^2\rho_i^2}{2}\left(\frac{m}{M}\right)\left(1+\frac{\Omega_{ce}^2}{\omega_{pe}^2}\right)+(\tfrac{1}{2}-\delta)\frac{1}{2(2\pi)^{1/2}k_2\rho_i}. \tag{3}$$

For $k_{1x}=-k_{2x}\gg k_{1y}, k_{2y}$, the above equation gives

$$k_1\rho_i\simeq(2k_0\rho_i^2/L)^{1/2}(m/M+\Omega_{ci}^2/\omega_{pi}^2)^{-1/2},$$

and the half-harmonic mode has positive energy according to (2) as does the low-frequency mode. Because the pump wave has negative energy, the parametric instability is an explosive one with energy feeding from the pump to the decay mode leading to the growth of all three. Experimentally one observes that the ion-cyclotron-wave amplitude is enhanced as the low-frequency modes grow.[2]

To evaluate the growth rate of this parametric instability, we note that because $e\varphi_0/T_i\ll 1$, the major nonlinearity in the parametric coupling is the electron's $\vec{E}\times\vec{B}$ drift in the oscillating pump field.

Let the potential of the pump field be $\varphi_0 = 2\varphi_0(x)\sin(k_0y - \omega_0 t)$ with $\omega_0 \approx \Omega_i$ and $k_0 \gg \partial \ln\varphi_0/\partial x$. Then the electron's $\vec{E}\times\vec{B}$ drift along the density gradient with $\tilde{v}_{0x} = cE_{0y}/B = 2v_{0x}\cos(k_0y - \omega_0 t)$ and $v_{0x} = -ck_0\varphi_0/B_0$ induces a density fluctuation

$$\tilde{n}_0(x,t) = -(2k_0c/B\omega_0)\varphi_0(x)(\partial\bar{n}_0/\partial x)\sin(k_0y - \omega_0 t),$$

where $\bar{n}_0(x) = n_0(x,t) - \tilde{n}(x,t)$ is the static equilibrium density. Let us now perturb this equilibrium of nonuniform magnetized plasma in a large-amplitude electrostatic wave with a perturbation potential $\tilde{\varphi}_1(x,t)$. Using the electron continuity equation, the Poisson equation with linear ion susceptibility, and neglecting the nonresonant Fourier components at $\omega+\omega_0$, $\vec{k}+\vec{k}_0$ (anti-Stokes), as well as higher-order terms at $\omega \pm l\omega_0$, $\vec{k}+l\vec{k}_0$ for $l \geq 2$, we obtain the following parametrically coupled equations for the components $\varphi_{\vec{k},\omega}$ and $\varphi_{\vec{k}-\vec{k}_0,\omega-\omega_0}$:

$$-i\omega\frac{k^2\epsilon(k,\omega)}{4\pi e}\varphi_{k,\omega} = ik_xv_0\frac{k_-^2\epsilon(\vec{k}_-,\omega_-)}{4\pi e}\varphi_- + ik_xv_0\frac{c}{B}\frac{\partial n_0}{\partial x}\left(\frac{k_0}{\omega_0} - \frac{k_y - k_0}{\omega-\omega_0}\right)\varphi_-, \tag{4}$$

$$-i(\omega-\omega_0)\frac{k_-^2\epsilon(k_-,\omega_-)}{4\pi e}\varphi_- = ik_xv_0\left\{\frac{k^2\epsilon(k,\omega)}{4\pi e} + \frac{c}{B}\frac{\partial n_0}{\partial x}\left(\frac{k_0}{\omega_0} - \frac{k_y}{\omega}\right)\right\}\varphi_{k,\omega}, \tag{5}$$

where $\epsilon(\vec{k},\omega)$ is the linear dielectric function, $\vec{k}_- = \vec{k}-\vec{k}_0$, $\omega_- = \omega - \omega_0$, and $\varphi_- \equiv \varphi_{\vec{k}-\vec{k}_0,\omega-\omega_0}$. Equations (4) and (5) are the basic equations for the parametric decay processes. Expanding $\epsilon(k,\omega)$ around the linear modes, $\epsilon(k,\omega) = (\partial\epsilon/\partial\omega)_{k_L,\omega_L}\delta\omega$ in Eqs. (4) and (5), we then obtain the growth rate for the parametric decay of the drift-cone mode:

$$\gamma \equiv i\delta\omega = \frac{|k_xv_0|v_{Di}}{(kk_-)^{1/2}\lambda_{Di}}\left[\left(\frac{k_0}{\omega_0} - \frac{k_y}{\omega}\right)\left(\frac{k_y - k_0}{\omega-\omega_0} - \frac{k_0}{\omega_0}\right)\right]^{1/2}\left[\left(\omega\frac{\partial\epsilon}{\partial\omega}\right)\left(\omega_-\frac{\partial\epsilon_-}{\partial\omega_-}\right)\right]^{-1/2}, \tag{6}$$

where $v_{Di} = cT_i/eBL$ and $\omega_-\partial\epsilon_-/\partial\omega_- = (\omega-\omega_0)\partial\epsilon_-/\partial(\omega-\omega_0)$ is negative. The growth rate for this explosive instability is readily evaluated from Eq. (6) by using the linear dispersion relations of the decay waves and noting $k_0/\omega_0 = 2k_y/\omega_1 \gg k_{2y}/\omega_2$,

$$\gamma = \frac{V_0}{\sqrt{2}\lambda_D}\left(\frac{k_0V_{Di}}{\omega_0}\right)\left(\frac{\omega_{pe}^2}{\Omega_{ce}^2}+1\right)^{-1} = \frac{k_0^2\rho_i^3}{2L}\left(\frac{e\varphi}{T_i}\right)\left(\frac{\omega_{pi}}{\Omega_{ci}}\right)\left(1+\frac{\omega_{pe}^2}{\Omega_{ce}^2}\right)^{-1}\Omega_i. \tag{7}$$

For the PR7 experiment, $e\varphi_0 \lesssim T_e \approx \frac{1}{40}T_i$, $k_0\rho_i \approx 6$, $\rho_i/L \approx \frac{1}{6}$. The corresponding growth rate is therefore of the order $0.1\Omega_i$ for $\omega_{pe}^2/\Omega_{ce}^2 \approx 1$, the low-density region where the instability is observed. The growth rate is reduced in the high-density region, which may account for its absence there.

Now we wish to show that, because the pump wave and decay modes propagate in the azimuthal direction, the axial asymmetry restricts the $\vec{k}$-matching condition to a limited region and the wave propagation out of this resonant cone can effectively stabilize this explosive instability. The threshold for the parametric instability in the presence of a linear spatial inhomogeneity in the direction of wave propagation is $\gamma_0^2/|V_1V_2K'| > 1$,[8] where γ_0 is the growth rate in an azimuthally symmetric system, $V_1 = \partial\omega_1/\partial k_{1y}$ and $V_2 = \partial\omega_2/\partial k_{2y}$ are the group velocities in the azimuthal direction of the two decay waves, $K' = d(k_0 - k_{1y} - k_{2y})/dy$ defines the width of the resonant zone due to azimuthal asymmetry. The group velocities are $V_1 = 4\Omega_i\rho_i^2/L\beta\pi^2$ for the half-harmonic mode and $V_2 = (\pi^2/4)V_1$ for the drift mode. Let the density scale length in the azimuthal direction be $L_y = (r^{-1}d\ln n/d\theta)^{-1}$. Then

$$dK/dy = (k_0/L_y)(\omega_{pe}^2/\Omega_{ce}^2+1)^{-1}(1 \pm \omega_2/\omega_1)$$

in which the first term dominates for $\omega_2 \ll \omega_1$. Substituting the above expressions and the growth rate, we then find threshold conditions for the parametric instability

$$\gamma_0^2/K'|V_1V_2| = \tfrac{1}{256}\pi k_0^2LL_y(e\varphi/T_i)^2 > 1. \tag{8}$$

For the $m=2$ flute asymmetry, $L_y = 2\pi L$. For typical parameters of the PR7 experiment,[1] the above condition is not satisfied and the explosive instability is therefore effectively stabilized by the azimuthal asymmetry, as the experiment demonstrated. At the maximum or minimum of the $m=2$ distortion, $dK/dy = 0$ and the stabilization condition[8] is $\gamma_0^2/|V_1V_2(K'')^{2/3}| < 1$, where $K'' = d^2K/dy^2 \approx k_0L_y^{-2}(1+\omega_{pe}^2/\Omega_{ce}^2)^{-1}$. The instability again is stabilized if

$$\frac{\pi}{(16)^2}\left(\frac{e\varphi}{T_i}\right)^2(k_0L)(k_0L_y)^{4/3}\left(1+\frac{\omega_{pe}^2}{\Omega_{ce}^2}\right)^{-1/3}<1.$$

When the plasma is relaxed to symmetry due to ΔB drift in a minimum-B configuration, this stabilization mechanism disappears as $L_y \to \infty$ and an explosive instability is expected. This explosive instability enhances both the pump energy and the decay-wave energy and therefore the overall level of turbulence, consistent with the observed increase of the level of fluctuation as the spectrum becomes turbulent. This turbulence accelerates ions stochastically, diffusing the ion into the loss cone. Enhanced ion loss therefore results. For a pump frequency at Ω_i, the decay into two half-harmonic modes is also possible; the growth rate, however, is weaker as the coupling is due to the electron-polarization drift rather than $\vec{E}\times\vec{B}$ drift. A warm plasma component also appears to have significant effect on the explosive instability and is the subject of a subsequent investigation

After this work was completed and submitted for publication, one of the authors (R.A.) presented this work in the U.S.S.R. and learned that Pastukhov[9] has studied the parametric decay into two modes: $\omega_1, \omega_2 \ll \Omega_i$. The decay channels being different, the resulting growth rate and stabilization condition also appear to be entirely different. Nor is the explosive nature of the instability and its consequences discussed in Ref. 9.

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Photoemission from Surface State of W(100): Evidence for $\vec{p}\cdot\vec{A}$ Mode of Photoionization

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The surface resonance peak is a prominent feature in the photoelectron spectrum of the clean W(100) surface when $h\nu = 16.8$ eV, but is not observed when $h\nu = 26.9$ eV. This indicates that the intensity of this peak is critically dependent upon the $\vec{p}\cdot\vec{A}$ matrix element, which should be large for photon energies below the tungsten plasmon energy of 23 eV, but small for photon energies above it. The consequences of this on the nature of the surface resonance are discussed.

In recent theoretical work, it has been recognized that the assumptions made in the calculation of cross sections in gas-phase photoelectron spectroscopy are not always valid when metal surfaces are considered.[1] The operator connecting initial and final states in the photoionization process has long been known from quantum electrodynamics to be $\vec{p}\cdot\vec{A}+\vec{A}\cdot\vec{p}$, where $\vec{A}$ is the vector potential of the photon and $\vec{p}$ is the quantum mechanical momentum operator.[2] In gas-phase photoelectron spectroscopy[1] the $\vec{p}\cdot\vec{A}$ term is generally neglected, since the electromagnetic field of the photon does not vary over distances much larger than a molecule (i.e., $\partial A_x/\partial x \approx 0$). The