Long-Time Tails and the Large-Eddy Behavior of a Randomly Stirred Fluid

Dieter Forster
Department of Physics, Temple University, Philadelphia, Pennsylvania 19122
and
David R. Nelson*
Department of Physics, Harvard University, Cambridge, Massachusetts 02138
and
Michael J. Stephen†
Department of Engineering and Applied Physics, Harvard University, Cambridge, Massachusetts 02138
(Received 22 January 1976)

The long-wavelength and low-frequency behavior of an incompressible, randomly stirred fluid is determined in d dimensions by renormalization-group arguments. A free-field fixed point, which describes conventional hydrodynamics, is stable for $d \geq 2$ although we find nontrivial corrections to the leading behavior. These corrections give rise to long-time tails in a fluid near thermal equilibrium. A nontrivial fixed point controls the behavior for $d<2$, which is determined to all orders in $\epsilon = 2-d$.

There has been considerable progress recently in the application of renormalization-group methods to the study of critical dynamics.[1] We apply these methods here to analyze the long-wavelength, low-frequency properties of a (noncritical) incompressible fluid subject to a random force. Analysis near a free-field fixed point $d>2$ produces the familiar long-time tails in the renormalized viscosity, and predicts new singularities at small wave numbers as well. Renormalization-group theory leads naturally to a unified treatment of these singularities, and provides a scaling description of the breakdown of hydrodynamics which occur for $d \leq 2$. These results pertain both to a fluid near thermal equilibrium and to the large eddies of a randomly stirred turbulent fluid.

We consider the dynamics of a system described by the Navier-Stokes equation:

$$\partial_t \vec{v} + \lambda \vec{v} \cdot \nabla \vec{v} = -\lambda \nabla P + \nu_0 \nabla^2 \vec{v} + \vec{f}, \tag{1a}$$

$$\nabla \cdot \vec{v} = 0, \tag{1b}$$

where $\vec{v}$ is the fluid velocity, P is the pressure, ν_0 is the (unrenormalized) viscosity coefficient, and λ is a perturbative parameter which will eventually be set equal to unity. The random force $\vec{f}$ is assumed to be a Gaussian random variable whose Fourier transform has a correlation function

$$\langle f_i(\vec{k}, \omega) f_j^*(\vec{k}', \omega') \rangle = 2D_0 k^2 \delta(\vec{k} - \vec{k}') \delta(\omega - \omega') D_{ij}(\vec{k}), \tag{2}$$

where $D_{ij}(k) = \delta_{ij} - k_i k_j / k^2$.

The above equations can be interpreted in at least two different ways: For a fluid near equilibrium at temperature T, $\vec{f}$ can be regarded as a noise field simulating the effects of the molecular degrees of freedom. The fluctation-dissipation theorem gives $D_0 = \nu_0 kT$. Alternatively, (1) can be regarded as a model of homogeneous, isotropic turbulence[2] with a Gaussian random, macroscopic stirring force generating a statistically steady state. This behavior of the random force holds for small k and ω provided that the spatial integral of the average force vanishes. In the region where (2) holds, the input of energy into the system is proportional to $D_0 k^2$. Since energy is dissipated at a rate proportional to $\nu_0 k^2$, we expect none of the cascade effects in this small-k region which are believed to characterize turbulence at large wave numbers.[3,4]

A quantity often studied in the theory of fluids is the velocity correlation tensor[3]

$$G_{ij}(k, \omega) = \langle v_i(\vec{k}, \omega) v_j^*(\vec{k}, \omega) \rangle. \tag{3}$$

In the present case this can be written in the form[5]

$$G_{ij}(k, \omega) = 2 \,\mathrm{Re}\{\chi D_{ij}(k) / [-i\omega + k^2 \nu(k, \omega)]\}, \tag{4}$$

where $\chi = D_0/\nu_0$ is the static susceptibility and $\nu(k, \omega)$ a kinematic viscosity. We will show that the dynamic renormalization group (DRG) drives the system to a fixed point where $\nu(k, \omega) = \nu$, a constant. Asymptotic corrections [corresponding to the $t^{-d/2}$ tail in the Green-Kubo integrand,[6,7] $\nu(k=0, t)$] result from the approach to the fixed point above two dimensions.

Also of interest is the spectral density function

$$E(k) = k^{d-1} \int_{-\infty}^{\infty} d\omega\, G_{ij}(\vec{k}, \omega), \tag{5}$$

the integral of which is proportional to the total kinetic energy of the fluid. $E(k)$ can be shown to behave like k^{d-1} with no correction terms in the regime where (2) holds.[8] However, deviations from (2) and from the assumed Gaussian behavior of the random force can lead to corrections to this result.[9]

The DRG procedure has been extensively discussed in Ref. 1. As usual, the Fourier modes will be cut off at large k:

$$v_i(\vec{r},t) = \int_{k<\Lambda} \frac{d^d k}{(2\pi)^d} \int \frac{d\omega}{2\pi} e^{i\vec{k}\cdot\vec{r}-i\omega t} v_i(\vec{k}, \omega). \tag{6}$$

The procedure consists of two parts: (a) Eliminate from (1) modes $v_i^{>}(\vec{k}, \omega)$ such that $\Lambda e^{-l} < k < \Lambda$ by averaging over $f_i^{>}(\vec{k}, \omega)$; and (b) rescale $\vec{k}, \omega$ and the remaining fields $v_i^{<}(\vec{k}, \omega)$, replacing $v_i^{<}(\vec{k}, \omega)$ by $\zeta v_i(ke^l, \omega e^{\alpha(l)})$, where $\alpha(l) \equiv \int_0^l dl' z(l')$. We choose the scale factor $\zeta = e^{ld/2+\alpha(l)}$ which ensures that the coefficient of ∂_t in (1) is unrenormalized.

As a result of step (a), Eq. (1a) takes the form

$$\partial_t \vec{v}^< + \lambda_a(\vec{v}^< \cdot \nabla)\vec{v}^< = -\lambda_a \nabla P^< + \nu_a \nabla^2 \vec{v}^< + \vec{f}^< + \vec{\psi}^<, \tag{7}$$

where $\vec{\psi}^<$ contains terms of higher order in $\vec{v}^<$ than those explicitly shown and also terms which contribute a $\vec{k}$, ω dependence to ν_a and λ_a. Step (b) then results in an equation like (1a), with extra terms generated from $\vec{\psi}^<$ and new coupling constants $\nu(l)$ and $\lambda(l)$. The Gaussian random force now obeys an equation similar to (2), but with a renormalized strength $D(l)$. It can be shown that the ratio $D(l)/\nu(l)$ is constant along a renormalization-group trajectory which is consistent with the fluctuation-dissipation theorem.[9]

Differential recursion relations for $\nu(l)$, $\lambda(l)$, and $D(l)$ are easily constructed near two dimensions. To second order in the reduced coupling

$$\bar{\lambda}(l) = D^{1/2}(l)\lambda(l)/\nu^{3/2}(l),$$

we find

$$d\nu/dl = (-2+z)\nu + \lambda^2 D/16\pi\nu^2, \tag{8a}$$

$$d\lambda/dl = (z-1-\tfrac{1}{2}d)\lambda, \tag{8b}$$

$$dD/dl = (-2+z)D + \lambda^2 D^2/16\pi\nu^3. \tag{8c}$$

For $d = 2-\epsilon$, $\bar{\lambda}(l)$ tends to a nontrivial fixed point of order $\epsilon^{1/2}$. When $d \geq 2$, $\bar{\lambda}(l)$ goes to a trivial fixed point $\bar{\lambda} = 0$. The terms $\vec{\psi}^<$ in (7) can be neglected to this order.

The calculation sketched above can be done more generally without the restriction $|\epsilon| \ll 1$. Requiring for convenience that $\nu(l)$ remain unrenormalized along a trajectory, which determines $z(l)$, we find that for *arbitrary* ϵ, the equations for $\bar{\lambda}(l)$ and $z(l)$ take the form:

$$d\bar{\lambda}/dl = [\tfrac{1}{2}\epsilon - \varphi(\bar{\lambda})]\bar{\lambda}, \tag{9a}$$

$$z(l) = 2 - \varphi(\bar{\lambda}). \tag{9b}$$

The function $\varphi(\bar{\lambda})$ is generated from self-energy diagrams and includes the effects of the term $\vec{\psi}^<$. The simplicity of these equations [note that the same function φ appears in both (9a) and (9b)] is a consequence of Galilean invariance. This requires that λ be renormalized only by a trivial multiplicative factor as shown in (8b).[10] We have verified this consequence of Galilean invariance to all orders in perturbation theory.[9] It is possible to argue on physical grounds that the function φ is positive, an assertion that is easily verified to $O(\bar{\lambda}^2)$ from (8).

Renormalization-group theory[11] leads to a homogeneity equation for $G_{ij}(\vec{k}, \omega, \bar{\lambda})$ valid for small $\vec{k}$ and ω:

$$G_{ij}(\vec{k}, \omega, \bar{\lambda}) = e^{\alpha(l)} G_{ij}(\vec{k}e^l, \omega e^{\alpha(l)}, \bar{\lambda}(l)), \tag{10a}$$

with

$$\alpha(l) = \int_0^l dl' z(l'). \tag{10b}$$

Similar relations are easily constructed for arbitrary velocity correlations. Defining a frequency-dependent viscosity at small k as the pole of (4), $i\omega = k^2\nu(\omega, \bar{\lambda})$, we have

$$\nu(\omega, \bar{\lambda}) = e^{2l-\alpha(l)} \nu(\omega e^{\alpha(l)}, \bar{\lambda}(l)). \tag{11}$$

For $d > 2$, $\bar{\lambda}(l) \to 0$ for large l and we have $z(l) \simeq 2$. The asymptotic theory for a fluid near equilibrium is conventional hydrodynamics. The result $z \simeq 2$ together with an analysis of the corrections to the asymptotic scaling behavior[12] leads to

$$\nu(\omega) = \nu(1 + C_d(i\omega/2\nu)^{(d-2)/2}). \tag{12}$$

This represents a long-time tail, $\nu(t) \sim t^{-d/2}$. Perturbation theory gives $C_3 = 7Di/120\pi\nu^3$.

For $d = 2$, $\bar{\lambda} = 0$ is still a stable fixed point, but $\bar{\lambda}(l)$ relaxes very slowly to its fixed-point value. This situation is analogous to what occurs for static isotropic spin systems in four dimensions.

From the above analysis we obtain

$$\nu(\omega) \sim [\ln(1/\omega)]^{1/2}. \tag{13}$$

This result has also been obtained by arguments based on renormalized perturbation theory and mode-coupling formulas.[13] Our derivation of (13) takes explicit account of vertex corrections, which are often neglected in mode-coupling analysis. We note that (13) is within the rigorous bounds proved by Wolynes.[14]

For $d<2$, a nontrivial fixed point becomes stable given by the solution of $\varphi(\bar{\lambda}^*)=\epsilon/2$. In this case $z\to z^*=2-\epsilon/2$ for large l, a result which holds to all orders in ϵ. A general result follows:

$$\nu(\omega)\sim(1/\omega)^{(2-d)/(2+d)}, \quad d<2, \tag{14}$$

which gives $\nu(t)\sim t^{-2d/(2+d)}$. A $t^{-2/3}$ asymptotic law in one dimension has been suggested by Pomeau and Rêsibois.[13]

Results such as (10) can also be applied to the large-eddy behavior of a randomly stirred fluid. Defining $\nu(k)$ as the renormalized k-dependent viscosity at $\omega=0$, we find, for $|\epsilon|$ small,

$$\nu(k)=\nu[1+(D/8\pi\nu^3\epsilon)(k^{-\epsilon}-\Lambda^{-\epsilon})]^{1/2}. \tag{15}$$

This expression goes smoothly into logarithmic behavior in two dimensions. In exactly $d=3$, perturbation theory gives $\nu[1-(D/32\pi\nu^3)(\frac{1}{3}+\frac{1}{4}\pi)k]$.

More generally the function $\nu(k,\omega)$ defined in Eq. (4) can be represented by a nontrivial scaling function for $d<2$. We find

$$\nu(k,\omega)\simeq\nu(2k)^{\epsilon/2}f((i\omega/2\nu)(2/k)^{z^*}), \tag{16}$$

where

$$\begin{aligned} f(x)&=1-\tfrac{1}{8}\epsilon-\tfrac{1}{2}\epsilon(1-\ln 2)x+\dots, \quad x\ll 1,\\ &=x^{1-2/z^*}-\tfrac{5}{12}\epsilon x^{1-4/z^*}+\dots, \quad x\gg 1, \end{aligned} \tag{17}$$

to first order in $\epsilon=2-d$. These results were obtained from a Feynman-graph expansion with $\bar{\lambda}^*=(8\pi\epsilon)^{1/2}+O(\epsilon^{3/2})$, its fixed-point value.

We can also present our results in the form of a dispersion relation, $\omega=\Omega(k,\bar{\lambda})$, for the hydrodynamic mode in a fluid near equilibrium. The scaling result is summarized by

$$\Omega(k,\bar{\lambda})=e^{-\alpha(l)}\Omega(ke^l,\bar{\lambda}(l)). \tag{18}$$

To leading order we obtain $\Omega\sim k^2(1+A_dk^{d-2})$ for $d>2$, $\Omega\sim k^2[\ln(1/k)]^{1/2}$ for $d=2$, and $\Omega\sim k^{1+d/2}$ for $d<2$.

We have disregarded the terms contained in $\vec{\psi}^<$ in Eq. (7). Such terms may already be present in the original model (1). It is easy to see that all higher-order terms (in $\vec{v}$ and ∇) are irrelevant compared with those proportional to λ. For example, the term $\nabla^4\vec{v}$ scales as $e^{(z-4)l}$ and can then be disregarded asymptotically. Irrelevant variables can contribute to higher-order terms in expansions such as (12), (13), and (14). Similar arguments apply to non-Gaussian contributions to the force-force correlation function.

It is interesting to speculate on the *large*-wave-vector behavior of correlations generated by (1). It is in this regime that the correlations are expected to exhibit highly nontrivial behavior characterized by a turbulent cascade.[3] The results presented here suggest that a perturbative renormalization-group approach would soon generate an intractably large effective coupling $\bar{\lambda}$ at these wave vectors. Qualitatively similar difficulties have been encountered in certain theories of strong interactions.[15]

This work was carried out independently at Temple and Harvard University. We wish to thank M. Droz, M. Green, and J. Gunton at Temple and P. C. Martin and E. Siggia for helpful discussions.

---

*Junior Fellow, Harvard Society of Fellows. Work supported in part by the National Science Foundation under Grant No. DMR 72-02977 A03.

†On leave from Rutgers University. Work supported in part by the Rutgers University Research Council and the National Science Foundation under Contract No. GH38458.

[1]B. I. Halperin, P. C. Hohenberg, and S. Ma, Phys. Rev. Lett. 29, 1548 (1972), and Phys. Rev. B 10, 137 (1974); B. I. Halperin, P. C. Hohenberg, and E. Siggia, Phys. Rev. Lett. 28, 548 (1974); S. Ma and G. Mazenko, Phys. Rev. Lett. 33, 1384 (1974), and Phys. Rev. B 11, 4077 (1975); R. Freedman and G. Mazenko, Phys. Rev. Lett. 34, 1575 (1975).

[2]See, e. g., R. H. Kraichnan, J. Fluid Mech. 5, 497 (1959); H. W. Wyld, Jr., Ann. Phys. (N.Y.) 14, 143 (1961); P. C. Martin, E. D. Siggia, and H. A. Rose, Phys. Rev. A 8, 423 (1973).

[3]S. A. Orszag, in *Hydrodynamics*, edited by R. Balian (Gordon and Breach, New York, 1974).

[4]There are interesting *infrared* cascade effects in two dimensions. See R. H. Kraichnan, Phys. Fluids 10, 1417 (1967). These will not occur in a regime where the force correlations behave as indicated in Eq. (2).

[5]L. P. Kadanoff and P. C. Martin, Ann. Phys. (N.Y.) 24, 419 (1963).

[6]B. Alder and T. Wainwright, Phys. Rev. A 1, 18 (1970).

[7]N. H. Ernst, E. H. Hauge, and J. M. J. Van Leeuwen, Phys. Rev. A 4, 2055 (1971), and Ark. Fys. Semin. Trondheim, No. 15 (1975).

[8]Equal-time correlations governed by (3) are Gaussian distributed. See S. F. Edwards and W. D. McComb,

J. Phys. A 2, 157 (1969).
[9]A detailed account of the calculations described here will be given elsewhere.
[10]Similar arguments based on rotational invariance have been used by B. I. Halperin, P. C. Hohenberg, and E. D. Siggia, Phys. Rev. B 13, 1299 (1976), in an analysis of planar spin antiferromagnets.
[11]K. G. Wilson and J. Kogut, Phys. Rep. 12C, 77 (1974); M. W. Fisher, Rev. Mod. Phys. 42, 597 (1974); S. K. Ma, Rev. Mod. Phys. 45, 589 (1973).
[12]F. Wegner, Phys. Rev. B 5, 4529 (1972).
[13]Y. Pomeau and P. Résibois, Phys. Rep. 19C, 64 (1975).
[14]P. G. Wolynes, Phys. Rev. A 11, 1700 (1975).
[15]See, e. g., S. Coleman, in *Properties of the Fundamental Interactions, 1971*, edited by A. Zichichi (Editrice Compositori, Bologna, Italy, 1972), Part A, p. 358; H. D. Politzer, Phys. Rep. 14C, 131 (1974).

Anomalous Transport and Stabilization of Collisionless Drift-Wave Instabilities*

W. W. Lee and H. Okuda
Plasma Physics Laboratory, Princeton University, Princeton, New Jersey 08540
(Received 10 November 1975)

The nonlinear evolution of collisionless drift-wave instabilities and the associated plasma transport have been studied extensively using particle-code simulations. It is found that the quasilinear decay of the density profile gives rise to the nonlinear saturation. The results also indicate that a new mechanism of wave absorption is responsible for the observed anomalous energy transport, which, in general, is larger than the corresponding particle diffusion and is also less sensitive to shear.

Low-frequency gradient-driven microinstabilities in a magnetically confined plasma have attracted wide-spread interest in recent years in view of the fact that the resulting enhanced plasma transport is detrimental to confinement.[1,2] While much of the theoretical work has been directed at obtaining relevant stability criteria for these modes,[2] their nonlinear behavior and the associated plasma transport processes are far less understood.[1,3,4] It is generally believed that particle-code simulations should play a key role in helping us to gain insight into these areas and should provide guidelines for the analytic work.

With that in mind, we have conducted extensive numerical studies on the drift-wave instability driven by the finite-Larmor-radius effects in a low-β collisionless plasma (universal mode), using a newly developed particle-simulation code.[5] In this Letter, we will report on the comparisons of our results with the existing linear theories.[6-8] Such comparisons so far are unavailable from laboratory experiments.[9,10] We will also present results concerning the nonlinear behavior of the instability with regard to the mechanisms for the nonlinear saturation and the anomalous plasma transport, and the scaling laws in the presence of shear. It is our opinion that these results will have an important influence on the future development of the nonlinear theory for the gradient-driven microinstabilities.

A $2\frac{1}{2}$ dimensional (x, y, v_x, v_y, v_z) bounded-plasma model capable of handling a nonuniform system has been developed for our purpose.[5] The system is uniform and periodic in y, and is nonuniform in x where the plasma is bounded between two conducting walls. The main magnetic field $\vec{B}_0$ is perpendicular to the inhomogeneous x direction with $B_{0z} \gg B_{0y}$. Particles reaching the walls are reflected so as not to produce sheath currents and other undesirable effects.[5]

Let us first present a linear theory pertaining to our model. For the case of the universal mode, the governing equation for the perturbed potential $\varphi = \tilde{\varphi}(x)\exp(ik_y y - i\omega t)$ can be written as[4]

$$\sum_\alpha \frac{1}{T_\alpha}\left[1 + \frac{\omega + \omega_\alpha^*}{\sqrt{2}\,k_\parallel v_{t\alpha}} I_0(\hat{b}_\alpha)e^{-\hat{b}_\alpha} Z\left(\frac{\omega}{\sqrt{2}\,k_\parallel v_{t\alpha}}\right)\right]\tilde{\varphi} = 0, \qquad \hat{b}_\alpha = \rho_\alpha^2(k_y^2 - \partial^2/\partial x^2) = b_\alpha - \rho_\alpha^2\,\partial^2/\partial x^2, \tag{1}$$

where α denotes the species, $v_{t\alpha} = (T_\alpha/m_\alpha)^{1/2}$, $\rho_\alpha = v_{t\alpha}/\omega_{c\alpha}$, $\omega^* = k_y T_e \kappa/m_i\omega_{ci}$, $\omega_e^* = -\omega^*$, $\omega_i^* = \omega^* T_i/T_e$, $\kappa = -(dn/dx)/n$, $L_n \equiv 1/|\kappa|$ is the density scale length, $k_\parallel = k_y\cos\theta$, where θ is the angle between $\vec{k}$ and $\vec{B}_0$, I_l is the Bessel function, and Z is the plasma dispersion function.[11] Expanding Eq. (1) in the