177 (1968).

[12]I. Iben, Jr., Ann. Phys. (N.Y.) 54, 164 (1969).

[13]J. W. Schopf and E. S. Barghoorn, Science 156, 508 (1967). There is other geological evidence that the oceans were not frozen (3–4) $\times 10^9$ yr ago, and that a temperature drop today of only 6% would suffice to freeze them—see W. L. Donn, B. D. Donn, and W. G. Valentine, Geol. Soc. Am. Bull. 76, 287 (1965).

[14]E. Teller, Phys. Rev. 73, 801 (1948). In fact, the oceans would have been boiling 3×10^9 yr ago if $m=-1$.

[15]A. Sandage, Quart. J. Roy. Astron. Soc. 13, 282 (1972).

Pseudopotentials of Estabrook and Wahlquist, the Geometry of Solitons, and the Theory of Connections

Robert Hermann*

Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts 02138

(Received 3 December 1975)

The prolongation structure of Wahlquist and Estabrook is interpreted as a connection. In this way, some geometric insight might be provided for the description of those nonlinear partial differential equations which admit soliton solutions. A new geometric property—linked to the existence of an SL(2,R) connection—is proved for the solutions of the Korteweg–de Vries equation.

Recent discovery that certain types of nonlinear partial differential equations in two independent variables possess special solutions ("solitons") with remarkable mathematical and physical properties has stimulated wide interest among physicists, engineers, and mathematicians. At this stage, one would like to know how general are these properties and whether they extend to equations with more than two independent variables. This seems to be a question of differential geometry; Wahlquist and Estabrook have made[1] very innovative and ingenious suggestions about the type of mathematical structures that might be involved. The purpose of this note is to show that part of their work may most naturally be interpreted in terms of the theory of connections.[2,3] I will deal only with the Korteweg–de Vries (K-dV) equation, but the methods are probably quite general. I will also only deal with the simplest "one-variable, quadratic" sort of what they call a "prolongation," but the theory of fiber bundles provides a basis for lifting this restriction.

I follow the notations in Wahlquist and Estabrook's paper.[1] The K-dV equation

$$u_t + u_{xxx} + 12uu_x = 0, \tag{1}$$

is to be solved for a function $(t,x)\to u(x,t)$ of two variables. Introduce a five-dimensional space, denoted as X, with variables (x,t,u,z,p) with

$$z = u_x; \quad p = z_x = u_{xx}. \tag{2}$$

On this space introduce the following second-degree differential forms[4]:

$$\begin{aligned} \alpha_1 &= du\wedge dt - z\,dx\wedge dt,\\ \alpha_2 &= dz\wedge dt - p\,dx\wedge dt,\\ \alpha_3 &= -du\wedge dx + dp\wedge dt + 12uz\,dx\wedge dt. \end{aligned} \tag{3}$$

Let I be the differential ideal[5] generated by α_1, α_2, and α_3. Each solution of (1) then is determined by a two-dimensional integral manifold[5] of I on which $dx\wedge dt$ is nonzero.

Let Y be a one-dimensional space, parametrized by a single variable y. Let G be the matrix group SL(2,R), the 2×2 real matrices of determinant one. Choose Y as a coset space of G, acting via linear fractional transformations. The Lie algebra, $\underline{G}$, of G then acts on Y as vector fields that are *quadratic* in y, i.e., that are of the form

$$(a+by+cy^2)\,\partial/\partial y, \tag{4}$$

with a, b, and c real numbers. Let

$$E = X\times Y.$$

Regard E as a fiber bundle[2] over X, with G as structure group, Y as fiber. A *connection*[2] for for this fiber bundle is then determined by a one-differential form ω on E of the following form:

$$\omega = \omega_0 + y\omega_1 + y^2\omega_2 + dy, \tag{5}$$

where ω_0, ω_1, and ω_2 are one-forms on X. Set

$$\Omega_0 = d\omega_0 - \omega_0 \wedge \omega_1,$$
$$\Omega_1 = d\omega_1 - 2\omega_0 \wedge \omega_2, \qquad (6)$$
$$\Omega_2 = d\omega_2 - 2\omega_1 \wedge \omega_2.$$

They are the *curvature forms* of the connection. *Definition:* The connection defined by the 1-form ω is said to be *associated with the differential equation (1)* if the following condition is satisfied:

$$\Omega_1,\ \Omega_2,\ \Omega_3 \in I. \qquad (7)$$

In order to give an important example where (7) is satisfied, we can define

$$\omega = dy + (2u + y^2 - \lambda)\,dx - 4[(u+\lambda)(2u+y^2-\lambda) - \tfrac{1}{2}p - zy]\,dt, \qquad (8)$$

(λ is a constant). It is now readily verified [e.g., from formula (46) of Ref. 1] that relation (7) is satisfied.

We can now deduce from these formulas a new property of solutions of the K-dV equation. Let S be an integral manifold of I, i.e., a two-dimensional submanifold of X on which the forms in I are zero. We denote the restriction of the forms ω_0, ω_1, ω_2 to S by ω_0', ω_1', ω_2'. Combining relations (6) and (7), we see that these forms satisfy the following relations:

$$d\omega_0' = \omega_0' \wedge \omega_1',$$
$$d\omega_1' = 2\omega_0' \wedge \omega_2', \qquad (9)$$
$$d\omega_2' = 2\omega_1' \wedge \omega_2'.$$

Equations (9) are the *structure equations* of the Lie group G. They imply that there is a map $S \to G$ such that the translation-invariant forms on G pull back to the ω! Thus, we see that there is a correspondence between solutions of Eqs. (1) and two-dimensional surfaces in G.

Wahlquist and Estabrook have also suggested how the Bäcklund transformation and the *linear* differential equations involved in the inverse-scattering technique may be obtained with their pseudopotentials. These concepts fit very rationally into the differential-geometric interpolation considered in this paper. (For example, a Bäcklund transformation is a mapping $X \times Y \to X \times Y$ which carries the ideal I into the ideal generated by I and the connection forms. The linear equations are basically the parallel-transport equations for linear connections induced in the associated linear vector bundles.) Extensive generalization and specialization are now feasible; further work is in progress.

I would like to thank Frank Estabrook and Hugo Wahlquist for bringing this problem to my attention and for instruction in their methods. I also thank Brian Doolin of NASA-Ames and Sheldon Glashow of Harvard University for inviting me to visit their respective institutions while this work was done.

*National Research Council Senior Research Associate at NASA-Ames Laboratory, where part of this work was done. Also supported in part by the National Science Foundation under Grant No. MPS75-20427.

[1]H. D. Wahlquist and F. B. Estabrook, J. Math. Phys. (N.Y.) 16, 1 (1975).

[2]S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry* (Wiley, New York, 1963), Vol. I.

[3]R. Hermann, *Gauge Fields and Cartan-Ehresmann Connections* (Math Sci Press, Brookline, Mass., 1975), Part A.

[4]For differential-geometry notation, see C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).

[5]D. Cartan, *Les Systèmes Différentielles Exterieurs et Leurs Applications Géométriques* (Hermann & Cie, Paris, France, 1945.