Phase Transitions and Continuous Symmetry Breaking*

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We announce a new method for proving the existence of phase transitions and of spontaneous symmetry breaking. This method is then applied to various models, including a multicomponent $[(\vec{\varphi}\cdot\vec{\varphi})^2]_3$ quantum field model and classical, isotropic Heisenberg model.

Phase transitions and spontaneous symmetry breaking are phenomena observed in macroscopic systems (mathematically, systems with an infinite number of degrees of freedom). Many current theories of the fundamental interactions involve as a central theoretical concept the possibility that continuous symmetries may be broken by the physical vacuum.

Because of the basic role that phase transitions and symmetry breaking play in physics there has been much work devoted to a rigorous analysis of these phenomena. Previous work in this field has relied on finding exact solutions of certain (two-dimensional or mean-field–type) models[1,2] or on the Peierls argument[3] and its extensions.[4] (Additional techniques have been developed for infinite-range interactions.[5]) None of the methods in Refs. 1–4 has been successful in proving that a continuous internal symmetry can be spontaneously broken in three or more dimensions. (It is well known that this cannot occur in one or two dimensions.[6])

Here we describe a new strategy for proving the existence of phase transitions in three or more dimensions and apply it to models in quantum field theory and statistical mechanics. Our methods not only serve to exhibit broken continuous symmetries but also prove the existence of phase transitions for systems with *no* internal symmetry. In principle our methods also apply to systems in which translational invariance is not present because of impurities. Full details and further applications will appear later.[7]

The central idea behind our strategy is contained in an *a priori* bound on the infrared singularity of the two-point function in momentum space. Let $F(k)$ be the Fourier transform of the two-point function $\langle S_i S_j\rangle$ [or $\langle\vec{\varphi}(x)\cdot\vec{\varphi}(y)\rangle$]: $F(k)$ is a positive distribution of the form

$$F(k)=\alpha\,\delta(k)+F^c(k). \tag{1}$$

For a symmetric theory where $\langle S_i\rangle=0$, α is the long-range order and $\alpha\neq0$ implies the coexistence of pure phases with spontaneous magnetization. In order to prove long-range order ($\alpha>0$) we need an upper bound on $F^c(k)$ (step A) and a lower bound on $\langle S_0S_0\rangle$ (step B). (A) The upper bound takes the form

$$0\leq F^c(k)\leq \text{const}/Jk^2, \tag{2}$$

where J denotes the strength of the nearest-neighbor coupling. In field theory J is determined by the commutation relation

$$[\vec{\varphi}_j(\vec{x},0),[\vec{\varphi}_l(\vec{y},0),H]]=-J\delta_{jl}\delta(\vec{x}-\vec{y}).$$

Here H denotes the Hamiltonian of the theory and φ_j is the jth component of the field $\vec{\varphi}$. (B) For suitable values of the parameters (temperature, coupling constants) we show

$$\langle S_0S_0\rangle\geq\gamma. \tag{3}$$

Taking the Fourier transform of (1) and combining it with (2) and (3) we conclude (in the case of three or more dimensions)

$$\alpha\geq\gamma-\text{const}\,J^{-1}, \tag{4}$$

which is positive for J sufficiently large. Note that in one or two dimensions the Fourier transform of k^{-2} is divergent; hence (4) does not apply. We shall see that step A is easy in quantum field theory whereas step B is easy in classical spin systems.

Next we illustrate our ideas in three models.

Model 1: $(\vec{\varphi}\cdot\vec{\varphi})^2$ quantum field model in three space-time dimensions.—We consider an N-component ($N=1,2,3$) scalar field $\vec{\varphi}$ with Hamiltonian density

$$H(\vec{\varphi}) = H_0(\vec{\varphi}) + \lambda(\vec{\varphi}\cdot\vec{\varphi})^2 - \sigma(\vec{\varphi}\cdot\vec{\varphi}) + \text{counterterms}. \tag{5}$$

Here $H_0(\vec{\varphi})$ is the Hamiltonian density of a free field of mass $m_0 > 0$. The counterterms involve vacuum energy and mass renormalization and are *independent* of σ. *No* field strength renormalization is required.

The vacuum (Wightman) state for this model has been rigorously constructed by several authors,[8] for all $\lambda > 0$, $m_0 > 0$, and σ. Since there is *no* field strength renormalization the Källen-Lehmann representation for the two-point function has the form

$$\langle\vec{\varphi}(x)\cdot\vec{\varphi}(y)\rangle = \alpha + \int d\rho(m^2)\int[d^3k/(2\pi)^3]e^{ik\cdot(x-y)}(k^2+m^2)^{-1}, \tag{6}$$

where $d\rho \geq 0$, and $\int d\rho(m^2) = N$ (the number of components of $\vec{\varphi}$), corresponding to $J=1$. See Ref. 7 for details. Equation (6) completes step A. To complete step B, consider the following:

$$\langle:\vec{\varphi}(0)\cdot\vec{\varphi}(0):\rangle \equiv \lim_{x\to 0}\{\langle\vec{\varphi}(x)\cdot\vec{\varphi}(0)\rangle - N\int[d^3k/(2\pi)^3]e^{ikx}k^{-2}\} = \alpha + \int d\rho(m^2)\int[d^3k/(2\pi)^3][(k^2+m^2)^{-1} - k^{-2}] \leq \alpha.$$

Hence, in order to prove that $\alpha > 0$ it now suffices to show that

$$\langle:\vec{\varphi}(0)\cdot\vec{\varphi}(0):\rangle > 0, \tag{7}$$

provided σ is large enough (for fixed λ and m_0). In Ref. 7 we use the functional integral to prove that for fixed λ and m_0

$$\langle:\vec{\varphi}(0)\cdot\vec{\varphi}(0):\rangle > 0,$$

for all σ greater than some finite σ_c. Our results establish continuous symmetry breaking and the existence of Goldstone bosons in this model.

Model 2: Classical Heisenberg model.—In this model $S_j = \vec{\sigma}_j$ is a classical N-component spin with

$$|S_j|^2 = \sum_{l=1}^{N}(\sigma_j^{\,l})^2 = 1.$$

The spins are coupled via the Hamiltonian

$$-\beta H = \sum_{|j-i|=1} J\vec{\sigma}_j\cdot\vec{\sigma}_i. \tag{8}$$

The infinite-volume Gibbs state is defined as a limit of periodic states. Step A is accomplished by using the transfer matrix T, and the commutator inequality which is the lattice analog of (3):

$$\tfrac{1}{2}[\sigma,[\sigma,T]] \leq (2J)^{-1}T,$$

with $\sigma = \vec{\sigma}_0\cdot\vec{a}$ and $\vec{a}$ a unit vector. Since $\langle\vec{\sigma}_0\cdot\vec{\sigma}_0\rangle = 1$, we have, in three or more dimensions,

$$1 \leq \alpha + \text{const}\,J^{-1},$$

where the constant is independent of J. Thus, for large J, α is positive. This gives a lower bound on the critical temperature. For the three-component three-dimensional model the bound is $\approx\frac{1}{13}$ of the value obtained by high-temperature series.

Model 3: Phase transitions for spin systems without symmetry.—In this model $S_j = \sigma_j$ is the classical one-component spin and H is as in (8). At each lattice site j there is a single-spin distribution

$$\exp(h\sigma_j)d\nu(\sigma_j),$$

where $d\nu$ is a positive measure on the real line that may *not* have *any symmetry*, but has the property that there exist $\delta > 0$ and $\epsilon > 0$ such that

$$\nu((-\infty,-\delta)) \geq \epsilon, \quad \nu((\delta,\infty)) \geq \epsilon.$$

Then, for J large enough, in three or more dimensions there is at least one value $h_c(J)$ at which $\langle\sigma_0\rangle(h)$ is discontinuous as a function of h. Hence there is a phase transition (at h_c). The proof is a variant of the strategy described above [step A is as in model 2; step B follows from simple estimates on the pressure as a function of J that are uniform in h; finally one shows that

$$\lim_{h\to+\infty}\langle\sigma_0\rangle(h) > \delta,$$

and

$$\lim_{h\to-\infty}\langle\sigma_0\rangle(h) < -\delta,$$

from which our assertions follow].

We have also found models similar to model 3 for which translational symmetry is globally broken and which phase transitions. We refer the reader to Ref. 7 for further details.

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[1]These include the free Bose gas [H. Araki and E. J. Woods, J. Mat. Phys. 4, 637 (1963)]; the solution of

the two-dimensional Ising model [L. Onsager, Phys. Rev. 65, 117 (1944)]; and Lieb's solution of the ice problem [see E. H. Lieb, in *Statistical Mechanics and Quantum Field Theory,* edited by C. DeWitt and R. Stora (Gordon and Breach, New York, 1971)]. See also T. H. Berlin and M. Kac, Phys. Rev. 86, 821 (1952).

[2]These include (among others) the BCS model of superconductivity—see N. N. Bogoliubov, Zh. Eksp. Teor. Fiz. 34, 58 (1958) [Sov. Phys. JETP 7, 41 (1958)]; W. Thirring and A. Wehrl, Commun. Math. Phys. 4, 301 (1966), and references given there—; and maser and laser models—see K. Hepp and E. H. Lieb, Ann. Phys. (N.Y.) 76, 360 (1972), and in *Constructive Quantum Field Theory,* edited by G. Velo and A. Wightman (Springer, Berlin, 1973).

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[4]There is an enormous literature on this subject beginning with R. Griffiths; see *Statistical Mechanics and Quantum Field Theory,* edited by C. DeWitt and R. Stora (Gordon and Breach, New York, 1971), and references given there.

[5]See F. Dyson, Commun. Math. Phys. 12, 91 (1969); R. Israel, to be published.

[6]See N. D. Mermin and H. Wagner, Phys. Rev. Lett. 17, 1133 (1966); N. D. Mermin, J. Math. Phys. (N.Y.) 8, 1061 (1967); H. Ezawa and J. A. Swieca, Commun. Math. Phys. 5, 330 (1967); S. Coleman, Commun. Math. Phys. 31, 259 (1973); R. L. Dobrushin and S. B. Shlosman, Commun. Math. Phys. 42, 31 (1975).

[7]J. Fröhlich, B. Simon, and T. Spencer, to be published.

[8]The original control of ultraviolet divergences and the "linear bound" were obtained by J. Glimm and A. Jaffe, Fortschr. Phys. 21, 327 (1973); and further developed by J. Feldman, Commun. Math. Phys. 37, 93 (1974). Completion of the construction of the theory for small coupling was obtained independently by J. Magnen and R. Seneor, to be published. The construction for arbitrary coupling is due to J. Feldman and K. Osterwalder, to be published; and J. Fröhlich, to be published. Useful additional bounds may be found in E. Seiler and B. Simon, to be published; and Y. Park, to be published.

Neutron Scattering from the Ferroelectric Fluctuations and Domain Walls of Lead Germanate

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Neutron scattering from the uniaxial ferroelectric $Pb_5Ge_3O_{11}$ shows a quasielastic component in addition to the soft ferroelectric mode. At low temperatures, this scattering increases but can be suppressed by an applied electric field and probably arises from scattering by static domain walls. This scattering also has a maximum near T_c which is largely field independent and which we suggest may arise from scattering by the walls of moving domains or clusters.

The neutron scattering from the fluctuations at several structural phase transitions has been found to consist of two components; one component has a frequency comparable with normal phonon frequencies but the other corresponds to fluctuations which decay on a much longer time scale. In some cases, Nb_3Sn for example,[1] the linear coupling between the order-parameter fluctuations and the phonon-density fluctuations provides a possible mechanism for the very slow fluctuations. In other cases, $SrTiO_3$ for example,[2] it is more difficult to account for the slow fluctuations. It has been proposed that they arise from (a) relaxations of the local order parameter near a defect,[1,3] (b) the motion of large clusters or domain walls (solitons) which are in dynamic equilibrium near T_c,[4] or (c) explicitly nonclassical fluctuation effects.[5] In this Letter we report on neutron-scattering measurements of the quasielastic ferroelectric fluctuations in lead germanate, $Pb_4Ge_3O_{11}$. This material undergoes uniaxial ferroelectric phase transition from a paraelectric phase of symmetry $P\bar{6}$ to a ferroelectric phase of symmetry $P3$.[6] In this case symmetry prohibits a linear coupling between the phonon-density fluctuations and the electric polarization fluctuations in the paraelectric phase. Furthermore, the theory of fluctuations in uniaxial ferroelectrics shows that $d=3$ is the marginal dimension for these materials so that static properties are expected to be described by classical exponents with the possibility of logarithmic correc-