297 (1974); A. N. Bloch, in *Energy and Charge Transfer in Organic Semiconductors*, edited by K. Masuda and M. Silver (Plenum, New York, 1974), p. 159; U. Bernstein, P. M. Chaikin, and P. Pincus, Phys. Rev. Lett. 34, 271 (1975).

[2]J. Ferraris, D. O. Cowan, V. Walatka, and J. H. Perlstein, J. Am. Chem. Soc. 95, 948 (1973).

[3]L. B. Coleman, M. J. Cohen, D. J. Sandman, F. G. Yamagashi, A. F. Garito, and A. J. Heeger, Solid State Commun. 12, 1125 (1973).

[4]Yaffa Tomkiewicz, B. A. Scott, L. J. Tao, and R. S. Title, Phys. Rev. Lett. 32, 1363 (1974).

[5]J. C. Scott, A. F. Garito, and A. J. Heeger, Phys. Rev. B 10, 3131 (1974).

[6]J. E. Gulley and J. F. Weiher, Phys. Rev. Lett. 34, 1061 (1975).

[7]W. M. Walsh, Jr., L. W. Rupp, Jr., D. E. Schafer, and G. A. Thomas, Bull. Am. Phys. Soc. 19, 296 (1974).

[8]F. Wudl, G. M. Smith, and E. J. Hufnagel, J. Chem. Soc. D 1970, 1453.

[9]M. Kinoshita and H. Akamatu, Nature (London) 207, 291 (1965).

[10]A. Carrington and A. D. McLachlan, *Introduction to Magnetic Resonance* (Harper and Row, New York, 1966).

[11]A similar approach was suggested independently by A. J. Berlinsky and J. F. Carolan.

[12]Yaffa Tomkiewicz, F. Mehran, D. C. Green, and B. A. Scott, Bull. Am Phys. Soc 19, 334 (1974).

[13]Yaffa Tomkiewicz, unpublished.

[14]Yaffa Tomkiewicz and E. M. Engler, to be published.

[15]Yaffa Tomkiewicz, E. M. Engler, and T. D. Schultz, Phys. Rev. Lett. 35, 456 (1975).

[16]P. A. Berger, D. J. Dahm, G. R. Johnson, M. G. Miles, and J. D. Wilson, Phys. Rev. B 12, 4085 (1975).

[17]The total susceptibility data appearing in this report are the data published in Ref. 5, and were kindly provided to us by J. C. Scott. We preferred these data to ours because the susceptibility tail, indicating the presence of impurities or imperfections, was absent.

[18]F. Denoyer, R. Comès, A. F. Garito, and A. J. Heeger, Phys. Rev. Lett. 35, 445 (1975).

[19]Using different arguments, T. D. Schultz and S. Etemad (to be published) have come to the same conclusion.

[20]S. Etemad, Phys. Rev. B (to be published).

[21]More detailed analysis should be possible with single-crystal measurements of (1) static susceptibility by R. M. Herman, R. A. Craven, and M. B. Salamon, to be published; and of (2) EPR susceptibility by Y. Tomkiewicz, A. R. Taranko, and J. B. Torrance, to be published.

[22]A. J. Berlinsky, J. F. Carolan, and L. Weiler, Solid State Commun. 15, 195 (1974).

[23]D. B. Tanner, C. S. Jacobsen, A. F. Garito, and A. J. Heeger, Phys. Rev. Lett. 32, 1301 (1974).

[24]M. J. Rice and S. Strassler, Solid State Commun. 13, 125 (1973).

[25]P. A. Lee, T. M. Rice, and P. W. Anderson, Phys. Rev. Lett. 31, 462 (1973).

[26]Y. Tomkiewicz, T. D. Schultz, E. M. Engler, A. R. Taranko, and A. N. Bloch, Bull. Am. Phys. Soc. 29, 287 (1976).

[27]H. Fukuyama, T. M. Rice, and C. M. Varma, Phys. Rev. Lett. 33, 305 (1974).

[28]B. R. Patton and L. J. Sham, Phys. Rev. Lett. 33, 638 (1974).

[29]P. M. Horn and D. S. Rimai, to be published.

[30]Similar conclusions were reached by Etemad (Ref. 20) and P. M. Chaikin, R. L. Greene, S. Etemad, and E. M. Engler, Phys. Rev. B 13, 1627 (1976).

[31]J. B. Torrance, B. A. Scott, and F. B. Kaufman, Solid State Commun. 17, 1369 (1975).

Neural Counting and Photon Counting in the Presence of Dead Time*

Malvin Carl Teich† and William J. McGill
Columbia University, New York, New York 10027
(Received 10 November 1975)

The usual stimulus-based neural counting model for audition is found to be mathematically identical to the well-known semiclassical formalism for photon counting. In particular, we explicitly demonstrate the equivalence of McGill's noncentral negative binomial distribution and Peřina's multimode confluent hypergeometric distribution for a coherent signal imbedded in chaotic noise. Dead-time corrections, important both in neural counting and in photon counting, are incorporated in a generalized form of this distribution. Some specific implications of these results are discussed.

In an attempt to explain the relative frequency of multiple occurrences of accidents in a factory population, Greenwood and Yule[1] in 1920 provided an important and remarkably simple generalization of the Poisson process. These authors assumed that although the probability of accident for a given worker follows the simple Poisson law, variation in individual proneness to accident causes the accident rate to vary from individual to individual in the population. They then calculated the overall probability of multiple accidents using certain plausible density functions for this individual variation. Their formalism has widespread applicability in a variety of disciplines, and has been studied intensively in recent years, although it is not widely understood that Greenwood and Yule originated this method. Such compound or doubly stochastic Poisson processes, as they are now called, occur when a continuous signal drives the mean rate of a Poisson distribution.

The purpose of this communication is threefold. First, we suggest that the stimulus-based neural counting model for audition proposed by McGill,[2] and the semiclassical description for the photon counting detection of light considered by Purcell[3] and Mandel,[4] are equivalent from a mathematical point of view and may be formally represented in terms of Greenwood and Yule's compound Poisson process. For both of these problems, the underlying Poisson behavior may arise from the occurrence of independent events or from the superposition of a large number of arbitrary stochastic point processes. For the particular neural counting model considered by McGill, the information-carrying continuous rate parameter is represented by the time-integrated intensity (or energy) of the stimulus; for photon counting it is the time-integrated intensity of the light that drives the mean rate of the basic Poisson process. The integration time is taken to be the observation period and, in both cases, the discrete counting statistics mimic the continuous-signal energy fluctuations. The identity described above implies that the body of research results available for semiclassical photon counting, corresponding to the usual normal ordering of creation and annihilation operators in a quantum-electrodynamic formalism,[5] is applicable to neural counting, and vice versa.

Second, we demonstrate the equivalence of the neural counting results of McGill[2] and the photon counting results of Peřina,[6,7] for a sinusoidal (coherent) signal imbedded in broad-band Gaussian (chaotic) noise of the same center frequency. Both of these calculations make use of Rice's well-known statistical analysis in the continuous signal domain.[8] Their simultaneous appearance in 1967 is quite remarkable and underscores the importance of current analytical work in physics and electrical engineering for seemingly unrelated studies involving neural information processing and psychophysics. A number of generalizations of this work were subsequently advanced[9,10]; in particular, an expression for the counting distribution for coherent and chaotic signal components with different mean frequencies was obtained by Peřina and Horák[9] in 1969.

Finally, we use the method of Cantor and coworkers[11,12] to incorporate the effects of fixed dead time (refractoriness) in this generalized counting distribution. The applicability to neural counting of the essential mathematical results obtained from this exercise cannot be advanced with utter assurance in the current state of our knowledge; nevertheless Teich, Matin, and Cantor[13] have recently shown that refractoriness can play an important role in visual information processing at the ganglion cell level. Interest in these equations from a photon-counting point of view stems from the deleterious effects of dead time on system channel capacity and error performance.

The superposition of a coherent and a chaotic signal provides a useful example for a number of reasons. In auditory neurophysiological and psychophysical research, and in photon counting and optical communications as well, it is a frequently used stimulus. Acoustically *and* electromagnetically, it is a simple matter to create a pure tone (sinusoidal signal) imbedded in Gaussian (chaotic) noise. An amplitude-stabilized laser operated in the usual regime well above threshold also obeys this model exceedingly well.[14,15] Furthermore, as we will show shortly, it is not difficult to extract from these superposition results a number of limiting cases that have been studied in great detail experimentally.

The mathematical equivalence of McGill's neural counting distribution and Peřina's photon counting result may be explicitly demonstrated as follows. Using the notational correspondences provided in Table I, we rewrite Eq. (22) of Ref. 2, for the probability $p(n, T)$ that exactly n counts will be recorded in the time interval T, as

$$p(n, T)=[1+(M/\langle n_0\rangle)]^{-n}[1+(\langle n_0\rangle/M)]^{-M}\exp(-\langle n_s\rangle M/\langle n_0\rangle)\times\sum_{k=0}^{\infty}\frac{1}{n!}\frac{(k+n+M-1)!}{k!(k+M-1)!}\left[\frac{\langle n_s\rangle M^2}{\langle n_0\rangle(M+\langle n_0\rangle)}\right]^k. \tag{1}$$

Factors not dependent on the variable k have been removed from the summation. The quantities $\langle n_s\rangle$ and $\langle n_0\rangle$ represent the average number of counts arising from the coherent and the chaotic components

TABLE I. Notational correspondences among various authors for counting distributions of the noncentral negative binomial form (first 3 columns) and of the simple negative binomial form (last 2 columns).

Present paper	McGill[a]	Peřina[b]	Greenwood and Yule[c]	Mandel[d]
n	j	n	n	n
T	T	T	...	T
$p(n,T)$	$p(j)$	$p(n)$	f_n	$p(n,T)$
M	ν	M	r	$k=T/\xi$
$\langle n_s\rangle$	aE_s	$\langle n_c\rangle$	...	...
$\langle n_0\rangle/M$	aN_0	$\langle n_T\rangle/M$	c^{-1}	$\bar{n}\xi/T$
k	i	k	...	...
$\langle n\rangle$	$E(j)$	$\langle n\rangle$	M	$\bar{n}$
$\langle(\Delta n)^2\rangle$	$\sigma^2(j)$	$\langle(\Delta n)^2\rangle$	μ_2	$\langle\Delta^2 n\rangle$
W	x	W	λ	E
$P(W)$	$f(x)$	$P(W)$	$f(\lambda)$	$P(E)$
$\langle W\rangle$	$E(x)$	$\langle W\rangle$	...	$\bar{E}$
$\langle(\Delta W)^2\rangle$	$\sigma^2(x)$	$\langle(\Delta W)^2\rangle$	...	$\langle\Delta^2 E\rangle$

[a]Ref. 2.
[b]Refs. 6, 7, 9, 16–18.
[c]Ref. 1.
[d]Ref. 4.

of the source, respectively. M is the number of modes ($M \geq 1$), or degrees of freedom, contributing to the noise; this has been discussed in Refs. 2, 6, and 10, and an explicit expression for M will be forthcoming shortly. Following McGill,[2] we refer to this as the noncentral negative binomial distribution.

To proceed, we need only note that the infinite summation in Eq. (1) may be expressed in terms of the confluent hypergeometric function ${}_1F_1$ through the identity[7]

$$ {}_1F_1(M+n, M; z) = \frac{(M-1)!}{(M+n-1)!}\sum_{k=0}^{\infty}\frac{(k+n+M-1)!}{k!(k+M-1)!}z^k, \tag{2}$$

where

$$ z \equiv \langle n_s\rangle M^2/[\langle n_0\rangle(M+\langle n_0\rangle)]. \tag{3}$$

Combining Eqs. (1)–(3) yields the counting distribution

$$ p(n,T) = \frac{(n+M-1)!}{n!(M-1)!}\left(1+\frac{M}{\langle n_0\rangle}\right)^{-n}\left(1+\frac{\langle n_0\rangle}{M}\right)^{-M}\exp\left(-\frac{\langle n_s\rangle M}{\langle n_0\rangle}\right){}_1F_1\left(n+M, M; \frac{\langle n_s\rangle M^2}{\langle n_0\rangle(M+\langle n_0\rangle)}\right) \tag{4}$$

which is identically Peřina's photon counting formula for fully polarized light.[7] This distribution may be (and often is) written equivalently[7,16] in terms of the generalized Laguerre polynomial $L_n^{M-1}(z)$:

$$ p(n,T) = \frac{1}{(n+M-1)!}\left(1+\frac{M}{\langle n_0\rangle}\right)^{-n}\left(1+\frac{\langle n_0\rangle}{M}\right)^{-M}\exp\left(-\frac{\langle n_s\rangle M}{M+\langle n_0\rangle}\right)L_n^{M-1}\left(-\frac{\langle n_s\rangle M^2}{\langle n_0\rangle(M+\langle n_0\rangle)}\right). \tag{5}$$

It is informative to examine a number of special cases of this distribution: (a) For $M=1$, Eqs. (4) and (5) reduce to the single-mode versions first obtained by Lachs[19] and by Glauber,[20] respectively. (b) For $\langle n_0\rangle \to 0$, corresponding to a pure coherent signal, the noncentral negative binomial distribution reduces to a simple Poisson. (c) For $\langle n_s\rangle \to 0$, corresponding to pure Gaussian noise, the *simple* negative binomial distribution is recovered. This latter distribution was explicitly obtained by Greenwood and Yule[1] and by Mandel.[4] Connection with the notation of these authors is also provided in Table I. We note that independent, additive, noninterfering noise (such as that arising from the spontaneous discharge, background noise, or dark current) is easily incorporated into the counting distribution represented in Eqs. (4) and (5) by forming a convolution sum with the appropriate noise count statistics (which are usually Poisson).

Equation (5), and therefore Eqs. (1) and (4) as well, are approximate. It has been obtained under a number of assumptions, including a uniform noise density, although this latter condition can be relaxed at the expense of increased complexity.[9,17] The number of degrees of freedom, M, is expressed in terms of the second-order degrees of coherence for chaotic and coherent components [$\gamma_0(\tau')$ and $\gamma_S(\tau')$, respectively] through the relation[16,17]

$$M = (\langle n_0\rangle + 2\langle n_s\rangle)/(\langle n_0\rangle y_1 + 2\langle n_s\rangle \bar{y}_1), \tag{6}$$

with

$$\begin{aligned} y_1 &= 2T^{-2}\int_0^T (T-\tau')|\gamma_0(\tau')|^2 d\tau', \\ \bar{y}_1 &= 2T^{-2}\int_0^T (T-\tau')\mathrm{Re}\{\gamma_0(\tau')\gamma_s^*(\tau')\} d\tau'. \end{aligned} \tag{7}$$

Peřina *et al.*[16] have recently compared the relatively simple approximate counting distribution discussed above with that calculated from Laxpati and Lachs's[21] exact closed-form recursion relation for arbitrary spectra. The approximation turns out to be best for rectangular and Gaussian spectra, and worst for Lorentzian spectra, but excellent in practically all cases.

Relatively simple analytic expressions for the generating functions and moments of the counting distribution [including the count mean $\langle n\rangle$ and the count variance $\langle(\Delta n)^2\rangle$] have also been obtained.[17] Furthermore, the integrated intensity distribution $P(W)$ and its various generating functions and moments, including the mean $\langle W\rangle$ and the variance $\langle(\Delta W)^2\rangle$, are well known[17] although we do not discuss these further here.

The superposition of coherent and chaotic signal components with different mean frequencies (ω_s and ω_0, respectively) leads to the counting distribution

$$p(n,T) = \left(1 + \frac{\langle n_0\rangle}{M}\right)^{-M} \exp\left(-\frac{\langle n_s\rangle[M + \langle n_0\rangle(1-\kappa^2)]}{M + \langle n_0\rangle}\right) \times \sum_{j=0}^{n} \frac{[\langle n_s\rangle(1-\kappa^2)]^{n-j}}{(n-j)!(j+M-1)!}\left(1 - \frac{M}{\langle n_0\rangle}\right)^{-j} L_j^{M-1}\left(-\frac{\langle n_s\rangle\kappa^2 M^2}{\langle n_0\rangle(M+\langle n_0\rangle)}\right), \tag{8}$$

where

$$\kappa = [\sin\tfrac{1}{2}(\omega_s - \omega_0)T]/[\tfrac{1}{2}(\omega_s-\omega_0)T]. \tag{9}$$

This result, obtained by Peřina and Horák,[9] represents a generalization of Eq. (5) applicable to the heterodyning of a chaotic source with a coherent local oscillator.[22,23] The accuracy of Eq. (8), which is also an approximation, has been shown to be as good as or better than that of Eq. (5) [$\kappa = 1$], which is excellent as indicated earlier.[16,18]

We now introduce refractoriness into Eq. (8). Dead-time effects have been considered from a general point of view by a number of authors including Feller,[24] Smith,[25] and Takács.[26] A comprehensive bibliography of papers in this field has recently appeared.[27] Most of this work has been structured within the framework of renewal theory, and a number of counting-distribution-limit theorems have been obtained. Parzen[28] has introduced the general *type-p* counter, which reduces to paralyzable and nonparalyzable behavior as special cases. He specifically deals with the renewal *counting* process, but primarily from a general viewpoint Cantor and Teich[11,12] extended the results of De Lotto, Manfredi, and Principi[29] and Bédard[30] to obtain dead-time–corrected counting distributions for sources of arbitrary statistics, under a variety of conditions. The nonparalyzable counter was assumed to exhibit a fixed dead time τ, and to be unblocked at the beginning of the counting interval T. The corrected distribution is simply expressible in terms of the uncorrected distribution and the ratio τ/T. Even small values of τ/T (~1%) alter the counting distribution markedly.[11-13]

Rewriting Eq. (7) of Ref. 11 to emphasize its dependence on the ratio τ/T, the expression for the fixed-dead-time–corrected counting distribution (in those cases for which the formula is valid) is

$$p(n,T,\tau/T) = \sum_{k=0}^{n}\langle p_k(n,T)\rangle_W - \sum_{k=0}^{n-1}\langle p_k(n-1,T)\rangle_W, \tag{10}$$

where

$$\langle p_k(n,T)\rangle_W = \langle (k!)^{-1}\{\langle n\rangle[1-(\tau/T)n]\}^k \exp\{-\langle n\rangle[1-(\tau/T)n]\}\rangle_W. \tag{11}$$

In particular, for the general superposition source discussed above, the quantity $\langle p_k(n,T)\rangle_W$ is obtained by using the following replacements in the right-hand side of Eq. (8): $\langle n_0\rangle \to \langle n_0\rangle[1-(\tau/T)n]$, $\langle n_s\rangle$

$\rightarrow \langle n_s \rangle [1-(\tau/T)n]$, and $n \rightarrow k$. When, in addition, the receiver structure and signaling format are specified, receiver performance (represented by the psychometric function) and mutual information (represented by the channel capacity) can be obtained from Eqs. (8), (10), and (11), and a detection law can be calculated in the presence of dead time. This same distribution will also be useful in visual psychophysics, e.g., when the stimulus is laser radiation. In neurophysiological contexts, Eq. (10) appears to provide a description for the observed firing statistics of individual auditory and visual neural fibers in certain cases. We have already shown that the special case $\{M \gg 1, \langle n_s \rangle = 0, \tau/T \sim 0.1\}$, which reduces to the fixed-dead-time-corrected Poisson distribution, provides a good representation for the maintained discharge in the cat's retinal ganglion cell.[13]

---

*A preliminary report of this work was presented at a Columbia University Special Seminar in April 1975 and at the Annual Meeting of the Society for Neuroscience in November 1975.

†Work supported by the John Simon Guggenheim Memorial Foundation, the National Institute of Mental Health, and the National Science Foundation.

[1]M. Greenwood and G. U. Yule, J. R. Stat. Soc., Ser. A 83, 255 (1920).

[2]W. J. McGill, J. Math. Psychol. 4, 351 (1967).

[3]E. M. Purcell, Nature (London) 178, 1449 (1956).

[4]L. Mandel, Proc. Phys. Soc., London 72, 1037 (1958), and 74, 233 (1959).

[5]R. J. Glauber, Phys. Rev. 130, 2529 (1963), and 131, 2766 (1963); P. L. Kelley and W. H. Kleiner, Phys. Rev. 136, A316 (1964).

[6]J. Peřina, Phys. Lett. 24A, 333 (1967).

[7]J. Peřina, Acta Univ. Palacki. Olomuc. Fac. Rerum Nat. 27, 227 (1968).

[8]S. O. Rice, Bell Syst. Tech. J. 24, 46 (1945).

[9]J. Peřina and R. Horák, J. Phys. A 2, 702 (1969).

[10]J. Peřina, V. Peřinová, and L. Mišta, Opt. Acta 19, 579 (1972).

[11]B. I. Cantor and M. C. Teich, J. Opt. Soc. Am. 65, 786 (1975).

[12]B. I. Cantor, L. Matin, and M. C. Teich, Appl. Opt. 14, 2819 (1975).

[13]M. C. Teich, L. Matin, and B. I. Cantor, unpublished.

[14]J. A. Armstrong and A. W. Smith, Phys. Rev. Lett. 14, 68 (1965); A. W. Smith and J. A. Armstrong, Phys. Rev. Lett. 16, 1169 (1966).

[15]P. J. Magill and R. P. Soni, Phys. Rev. Lett. 16, 911 (1966).

[16]J. Peřina, V. Peřinová, G. Lachs, and Z. Braunerová, Czech. J. Phys. B23, 1008 (1973).

[17]J. Peřina, *Coherence of Light* (Van Nostrand-Reinhold, London, 1972).

[18]R. Horák, L. Mišta, and J. Peřina, Czech. J. Phys. B 21, 614 (1971).

[19]G. Lachs, Phys. Rev. 138, B1012 (1965).

[20]R. J. Glauber, in *Physics of Quantum Electronics*, edited by P. L. Kelley, B. Lax, and P. E. Tannenwald (McGraw-Hill, New York, 1966), pp. 788–811.

[21]S. R. Laxpati and G. Lachs, J. Appl. Phys. 43, 4773 (1972).

[22]M. C. Teich and R. Yen, J. Appl. Phys. 43, 2480 (1972).

[23]E. Jakeman, C. J. Oliver, and E. R. Pike, Adv. Phys. 24, 349 (1975).

[24]W. Feller, in *Studies and Essays: A Volume for the Anniversary of Courant* (Wiley-Interscience, New York, 1948), pp. 105–115.

[25]W. L. Smith, J. R. Stat. Soc., Ser. B 20, 243 (1958).

[26]L. Takács, Ann. Math. Stat. 29, 1257 (1958).

[27]"Bibliography on Dead Time Effects," Bureau International des Poids et Mesures, Sèvres, France, Report No. BIPM-75/6, 1975, edited by J. W. Müller (unpublished).

[28]E. Parzen, *Stochastic Processes* (Holden-Day, San Francisco, 1962), pp. 118–186.

[29]I. DeLotto, P. F. Manfredi, and P. Principi, Energ. Nucl. (Milan) 11, 557, 599 (1964).

[30]G. Bédard, Proc. Phys. Soc., London 90, 131 (1967).