velocity lower than the electron thermal velocity may not be excited in the trapped-electron regime in large toroidal devices.

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Coupling Saturation in the Nonlinear Theory of Parametric Decay Instabilities

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A new saturation mechanism for the nonlinear stage of parametric decay instabilities arises from pump-induced cross-correlations between Fourier modes. These correlations result in a renormalization—usually a reduction—of the mode coupling coefficients. Coupling saturation arises from the same order of approximation as the familiar induced-scattering nonlinearity. When applied to the $2\omega_{pe}$ decay instability and the electron-ion decay instability, coupling saturation greatly reduces the turbulent wave energy compared to earlier calculations.

To interpret recent laser and microwave experiments using intense pump waves, a vigorous effort has been made to understand the nonlinear behavior of parametric instabilities. This Letter describes a new and important contribution to the formulation of a more complete analytic theory which takes into account the mode correlations unique to parametric decay instabilities. Early work concentrated on the nonlinear modification of the wave damping.[1] These calculations were essentially an application of the usual formulation of weak turbulence.[2] Here, we show that the turbulence can modify the mode-mode coupling coefficients, as well as the wave frequency and damping, reducing the coupling to the pump. This new effect arises from correlations outside the random-phase approximation (RPA) which are induced by the parametric coupling, yet we show that they arise from the same approximation leading to the familiar induced-scattering type of nonlinearity.

For the sake of easy illustration, we consider only the purely electrostatic case. If we limit ourselves to terms quadratic in the electric field E in the nonlinear Poisson's equation, we have

$$\epsilon(k)E(k)=\frac{1}{2}\int\frac{d^4k'}{(2\pi)^4}\int\frac{d^4k''}{(2\pi)^4}\chi(k,k',k'')E(k')E(k'')(2\pi)^4\delta^4(k-k'-k''), \tag{1}$$

where ϵ is the linear dielectric function for a homogeneous, isotropic plasma, and χ is the lowest-order nonlinear susceptibility.[2] We emply the notation $(\vec{k},\omega)=k$. As is well known, a monochromatic pump wave,

$$E_0(k)=E_0(2\pi)^4[\delta^4(k-k_0)+\delta^4(k+k_0)],$$

linearly excites a pair of growing decay modes $(\vec{k}_1,\omega_1)$ and $(\vec{k}_2,\omega_2)$ where $\omega_0=\omega_1+\omega_2$ and $\vec{k}_0=\vec{k}_1+\vec{k}_2$ for those cases where we may neglect anti-Stokes coupling.[3] We assume then that nonlinearly the saturated spectrum is composed of a collection of paired waves $\vec{k}_{n_1},\omega_{n_1}$ and $\vec{k}_{n_2},\omega_{n_2}$ where

$$k_0=k_{n_1}+k_{n_2}. \tag{2}$$

This pairing, or correlation, exists in the linear theory. It is a direct result of the sinusoidal modulation of the system by the coherent pump in time and space which couples together modes whose frequencies and wave vectors differ by integral multiples of ω_0 and $\vec{k}_0$, respectively. Since this coupling arises from the pump-imposed *symmetry*, it must also exist in the nonlinear theory. In a more detailed and rigorous treatment,[4] we examine the response of the nonlinear or turbulent plasma to a small perturbation in the presence of the coherent pump modulation and show that the dispersion relation to be derived below follows from the marginal stability requirement of the nonlinear system. Therefore, we postulate that the total field satisfying Eq. (1) may be written

$$E(k)=E_0(k)+\sum_{s=1}^{2}\sum_{n_s}(2\pi)^4[E_{n_s}\delta^4(k-k_{n_s})+E_{n_s}{}^*\delta^4(k+k_{n_s})]+(\vec{k}_{n_s}\rightarrow-\vec{k}_{n_s})+\delta E(k). \tag{3}$$

Here, E_{n_1} and E_{n_2} are the complex amplitudes of the nth correlated pair associated with a given frequency ω_{n_1} or ω_{n_2}, and $\delta E(k)$ denotes the presumably weak contribution to the field from nonresonant beats of the excited pairs.

If we substitute (3) into (1), we obtain a set of coupled equations for E_{n_1}, E_{n_2}, and δE. To lowest (linear) order in δE, we find for E_{n_1}

$$\begin{aligned}\epsilon(k_{n_1})E_{n_1}&=\chi(k_{n_1},k_0,-k_{n_2})E_0E_{n_2}{}^*\\&+\frac{1}{(2\pi)^4\delta^4(0)}\sum_s\sum_{n'}[\chi(k_{n_1},k_{n_s'},k_{n_1}-k_{n_s'})E_{n_s'}\delta E(k_{n_1}-k_{n_s'})+(\vec{k}_{n_s'}\rightarrow-\vec{k}_{n_s'})].\end{aligned} \tag{4}$$

A similar equation can be written for E_{n_2}. The beat terms δE are given by

$$\begin{aligned}&\epsilon(k_{n_1}-k_{n_1'})\delta E(k_{n_1}-k_{n_1'})\\&\quad=(2\pi)^4\delta^4(0)[\chi(k_{n_1}-k_{n_1'},k_{n_1},-k_{n_1'})E_{n_1}E_{n_1'}{}^*+\chi(k_{n_1}-k_{n_1'},k_{n_2'},-k_{n_2})E_{n_2}{}^*E_{n_2'}],\end{aligned} \tag{5a}$$

and

$$\begin{aligned}&\epsilon(k_{n_1}-k_{n_2'})\delta E(k_{n_1}-k_{n_2'})\\&\quad=(2\pi)^4\delta^4(0)[\chi(k_{n_1}-k_{n_2'},k_{n_1},-k_{n_2'})E_{n_2'}{}^*E_{n_1}+\chi(k_{n_1}-k_{n_2'},-k_{n_2},k_{n_1'})E_{n_1'}E_{n_2}{}^*],\end{aligned} \tag{5b}$$

where we have used $k_{n_1}-k_{n_1'}=-k_{n_2}+k_{n_2'}$ which follows from (2). We will show that all the nonlinear terms in (4) and the corresponding equation for E_{n_2} are consistent with a renormalized set of coupled equations,

$$\tilde{\epsilon}(k_{n_1})E_{n_1}=\tilde{\chi}(k_{n_1},k_0,-k_{n_2})E_0E_{n_2}{}^*, \tag{6a}$$

$$\tilde{\epsilon}(k_{n_2})E_{n_2}=\tilde{\chi}(k_{n_2},k_0,-k_{n_1})E_0E_{n_1}{}^*, \tag{6b}$$

a result identical *in form* with that derived from the linear theory, except that the tilded quantities are now functions of the field intensities $|E_{n_1}|^2$ and $|E_{n_2}|^2$.

If we substitute the expressions of (5) for δE into (4), we find terms either proportional to E_{n_1}, and therefore contributing to $\tilde{\epsilon}$, comparing with (6a), or terms proportional to $E_{n_2}{}^*$ contributing to the right-hand side of (6b). Consideration of terms proportional to E_{n_1} leads to

$$\tilde{\epsilon}(k_{n_1})=\epsilon(k_{n_1})-\sum_s\sum_{n_s'}\left[\frac{\chi(k_{n_1},k_{n_s'},k_{n_1}-k_{n_s'})\chi(k_{n_1}-k_{n_s'},-k_{n_s'},k_{n_1})|E_{n_s'}|^2}{\epsilon(k_{n_1}-k_{n_s'})}\right]. \tag{7}$$

The terms proportional to $E_{n_2}{}^*$ occur proportional to products such as $E_{n_1'}E_{n_2'}E_{n_2}{}^*$. We may eliminate $E_{n_1'}$, say in favor of $E_{n_2'}{}^*$, by using the *Ansatz* (6a) which allows us to write for the renormalized $\tilde{\chi}$ in (6a)

$$\tilde{\chi}(k_{n_1},k_0,-k_{n_2}) = \chi(k_{n_1},k_0,-k_{n_2}) + \sum_{n_2'} \left\{ \left[\frac{\chi(k_{n_1},k_{n_1'},k_{n_1}-k_{n_1'})\chi(k_{n_1}-k_{n_1'},k_{n_2'},-k_{n_2})}{\epsilon(k_{n_1}-k_{n_1'})} + (k_{n_2'} \rightleftharpoons k_{n_1'}) \right] \times \frac{\tilde{\chi}(k_{n_1'},k_0,-k_{n_2'})|E_{n_2'}|^2}{\tilde{\epsilon}(k_{n_1'})} \right\}. \tag{8}$$

In (6b), $\tilde{\epsilon}(k_{n_2})$ is obtained from (7) by $k_{n_1} \to k_{n_2}$, whereas for the renormalized coupling coefficient

$$\tilde{\chi}(k_{n_2},k_0,-k_{n_1}) = \chi(k_{n_2},k_0,-k_{n_1}) + \sum_{n_2'} \left\{ \left[\frac{\chi(k_{n_2},k_{n_2'},k_{n_2}-k_{n_2'})\chi(k_{n_2}-k_{n_2'},k_{n_1'},-k_{n_1})}{\epsilon(k_{n_2}-k_{n_2'})} + (k_{n_2'} \rightleftharpoons k_{n_1'}) \right] \frac{\tilde{\chi}(k_{n_1'},k_0,-k_{n_2'})}{\tilde{\epsilon}(k_{n_1'})} |E_{n_2'}|^2 \right\}. \tag{9}$$

Equations (6), (7), (8), and (9) form a complete set for the determination of $|E_{n_s}|^2$. The nonlinear terms of $\tilde{\epsilon}$ are familiar from the usual form of weak turbulence theory, resulting in the modification of wave damping and frequency.[1] The contribution to $\mathrm{Im}\tilde{\epsilon}(\vec{k}_n, \omega_n)$ from the beating of high-frequency waves, $|E_{n'}|^2$, includes the process of induced scattering from ion fluctuations.[1,2] The nonlinear terms of $\tilde{\chi}$, on the other hand, are novel for several reasons. (1) They change the *coupling to the pump* of the excited fields. In many situations of interest, we find that $\tilde{\chi}$ is significantly reduced from its linear value, leading to saturated energies substantially lower than those from just $\tilde{\epsilon}$. (2) From the derivation above it is clear that these terms result from the correlated product $E_{n_1}E_{n_2}$. In the RPA, this correlation vanishes; however, it is intrinsic to the parametric coupling that this correlation is strong. (3) We see that these terms lead to an integral equation for $\tilde{\chi}$ as compared with $\tilde{\epsilon}$ where the nonlinearity is simply additive.

The same results will be derived from a more complete and rigorous formulation of weak turbulence theory in a longer publication.[4] It is important to emphasize that the coupling saturation (CS) is *distinct* from pump depletion. In the present work, the pump parameters are held *fixed.*

We now consider the nonlinear saturation of the electron-electron decay instability (EED or $2\omega_{pe}$) to illustrate the relative significance of the CS mechanism. Pustovalov, Silin, and Tikhonchuk (PST)[5] have recently calculated the nonlinear saturation of this instability, taking as the dominant saturation mechanism spreading of the spectrum in angle due to nonlinear induced scattering of waves from ions. We merely take over the results of their calculation showing how the CS mechanism enters. The solvability condition of Eq. (6) is equivalent to

$$(\omega_n - \tilde{\omega}_1 + i\tilde{\gamma}_1)(\omega_0 - \omega_n - \tilde{\omega}_2 + i\tilde{\gamma}_2) + \frac{\tilde{\chi}(k_n,k_0,k_n-k_0)\tilde{\chi}^*(k_0-k_n,k_0,-k_n)}{(\partial/\partial\omega_n)\mathrm{Re}\epsilon(k_n)(\partial/\partial\omega_n)\mathrm{Re}\epsilon(k_0-k_n)} E_0{}^2 = 0, \tag{10}$$

where we assume that the frequency, ω_n, is near one of the roots of $\tilde{\epsilon}(k)$ and $\tilde{\epsilon}(k-k_0)$, a condition which allows us to use resonance approximations for the ϵ's. In Eq. (10), $\tilde{\omega}_1 = \omega_l(\vec{k})$ and $\tilde{\omega}_2 = \omega_l(\vec{k}_0 - \vec{k})$ are the Langmuir frequencies for the decay plasmons (the nonlinear shift in the frequencies is small), and since $|\vec{k}_0 - \vec{k}| \simeq |\vec{k}|$, the corresponding damping rates are equal: $\tilde{\gamma}_1 \simeq \tilde{\gamma}_2 = \tilde{\gamma}_l$, where $\tilde{\gamma}_l$ includes the nonlinear enhancement.[5]

From the real and imaginary parts of Eq. (10), we have

$$\omega_n = \tfrac{1}{2}\omega_0 + \tfrac{1}{2}[\omega_l(\vec{k}_n) - \omega_l(\vec{k}_0 - \vec{k}_n)], \tag{11}$$

and the marginal stability condition

$$\tilde{\gamma}_l = [\tilde{\Gamma}^2 - (\Delta\omega/2)^2]^{1/2}, \tag{12}$$

where the coupling parameter $\tilde{\Gamma}^2$ is the last term in Eq. (10), and $\Delta\omega = \omega_l(\vec{k}) + \omega_l(\vec{k}_0 - \vec{k}) - \omega_0$. The solution of (12) obtained by PST with $\tilde{\Gamma}^2 = \Gamma^2$, i.e., in the absence of CS, is localized in k about k_m where $\Delta\omega(k_m) = 0$, but spread in angle about the fastest growing mode direction by nonlinear induced scattering from ions. They obtain, for the total energy normalized to the particle energy density,

$$W=\sum_{s=1,2}\sum_{n_s}|E_{n_s}|^2(4\pi n\theta_e)^{-1}=\rho(P^{1/2}-1)^2/P^{1/2}, \tag{13}$$

where

$$\rho=\frac{243}{\pi\sqrt{2}}\frac{\gamma_l}{\omega_{pe}}\frac{m_i}{m_e}\frac{v_e^2}{c^2}\left(1+\frac{T_i}{T_e}\right),$$

and $P=\Gamma^2/\gamma_l^2$, the ratio of the pump energy to the threshold energy. To include CS, we turn to Eq. (8). The susceptibilities in (8) are given by[2]

$$\chi(k_0-k,k_0-k',k'-k)=\chi(k'-k,k',-k)=\frac{i}{|\vec{k}-\vec{k}'|}\frac{e}{m}\frac{\hat{k}\cdot\hat{k}'}{v_e^2}. \tag{14}$$

From (8), we have, for $\tilde{\chi}_-(\vec{k})\equiv\tilde{\chi}(k_0-k,k_0,-k)$,

$$\tilde{\chi}_-(\vec{k})=\chi(\vec{k})-\frac{e^2k_{D_e}^2}{m^2\omega_{pe}^3(1+T_e/T_i)}\int\frac{d^3k'}{(2\pi)^3}\frac{(\hat{k}\cdot\hat{k}')^2|E(\vec{k}')|^2\tilde{\chi}_-(\vec{k}')}{\Delta\omega(\vec{k}')-2i\gamma_l(\vec{k}')}, \tag{15}$$

where we have taken the second term in (8) equal to the first since $|\vec{k}_0-\vec{k}|\simeq|\vec{k}|$, employed a resonant approximation for $\bar{\epsilon}(k-k_0)$ as in Eq. (10), and used the static limit for $|\vec{k}-\vec{k}'|^2\epsilon(\vec{k}-\vec{k}',\omega-\omega')\simeq k_{D_e}^2+k_{D_i}^2$. Since the spectrum, as derived by PST, is narrowly confined to $k'\approx k_m$, and if we assume that $\tilde{\chi}_-$ is slowly varying in angle compared to variation of $|E|^2$, we find that $\tilde{\chi}_-$ for the most unstable vector $\vec{k}_m$ is given by

$$\tilde{\chi}_-(\vec{k}_m)\simeq\chi(\vec{k}_m)\left[1+\frac{i\omega_{pe}W}{2\tilde{\gamma}_l(\vec{k}_m)(1+T_e/T_i)}\right]^{-1}. \tag{16}$$

A similar calculation for $\tilde{\chi}(k,k_0,k-k_0)|_{\vec{k}=\vec{k}_m}=\tilde{\chi}_+$ in (9) yields $\tilde{\chi}_+=\tilde{\chi}_-$. It follows from (12) that when $\Delta\omega=0$, $\tilde{\gamma}_l=\tilde{\Gamma}$. If we combine this with (16) and identify $\tilde{\Gamma}^2$ as the last term in (10), we find

$$\tilde{P}=\frac{\tilde{\Gamma}^2(\vec{k}_m)}{\gamma_l^2}=P\left[1+\frac{\omega_{pe}^2W^2}{4\gamma_l^2\tilde{P}(1+T_e/T_i)^2}\right]^{-1}\text{ or }\tilde{P}=P-\frac{1}{4}\frac{\omega_{pe}^2}{\gamma_l^2}\frac{W^2}{(1+T_e/T_i)^2},$$

which we now use to replace P in Eq. (13) to obtain W.

To solve the nonlinear equation for W resulting from this substitution we note that $\rho/W\gg1$, generally, and we find

$$W=2(\gamma_l/\omega_{pe})(P-1)^{1/2}(1+T_e/T_i). \tag{17}$$

Comparing with the value for $W=W_{\mathrm{PST}}$ in the absence of CS we have $W/W_{\mathrm{PST}}\sim\gamma_l/\omega_{pe}\rho\ll1$, if $P\gg1$. Thus CS greatly reduces the excited wave energy in this case. We do not regard (17) as a complete formula for the wave energy, however, since the resulting spectrum localized near k_m is probably unstable to decay into plasmons and ion waves which spreads the plasmon spectrum to lower values of k. The point we wish to make here is that CS greatly reduces the wave energy predicted by previous theories which used only induced scattering from ions as the saturation mechanism.

We will show elsewhere[4] that in certain cases (e.g., $T_e\gg T_i$), CS plays a dominant role in the saturation of the electron-ion decay instability.

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