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Renormalization of the Nonlinear σ Model in $2+\epsilon$ Dimensions—Application to the Heisenberg Ferromagnets

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The nonlinear σ model is renormalizable and asymptotically free in two dimensions. We show here how to construct this model in $2+\epsilon$ dimensions. Renormalization-group equations follow and exhibit a nontrivial uv stable fixed point, which corresponds in the language of Heisenberg ferromagnets to a critical point. The existence of systematic expansions in powers of $d-2$ follows from this analysis. In the presence of a source information about long-distance behavior above the critical coupling constant is obtainable.

Recently Migdal[1] and Polyakov[2] have discussed the behavior near two dimensions of an n-component scalar field coupled through an $O(n)$-invariant interaction.[3] We shall show how to recover and generalize their results using standard field-theoretical methods. For this purpose we shall construct a renormalizable nonlinear σ model in $2+\epsilon$ dimensions as a power series in ϵ. The main feature of this construction is that the resulting theory is, as noted by Polyakov, asymptotically free in two dimensions.[3] The same considerations would apply to any such situation, as for instance the Thirring model with internal symmetries, or non-Abelian gauge theories in $4+\epsilon$ dimensions.

For statistical mechanics the nonlinear σ model describes the infrared properties of the n-component classical Heisenberg ferromagnet as obtained from the low-temperature expansion. A renormalization-group equation similar to those derived around four dimensions[4] holds thus for these systems in $2+\epsilon$ dimensions. Its integration describes the critical scaling behavior with universal parameters given as ϵ series. The same Heisenberg system may thus be described above two dimensions by two different field theories, namely the linear and the nonlinear σ models, which are therefore presumably identical in the neighborhood of the fixed point.

Renormalization of the nonlinear σ model in $2+\epsilon$ dimensions.—The theory will be defined order by order in ϵ. The generating functional of the Green's functions in Euclidean space is written in terms of a σ field and $n-1$ Goldstone fields $\vec{\pi}$:

$$Z=\int[d\sigma\, d^{n-1}\pi]\prod\delta(\sigma^2+\vec{\pi}^2-1)\exp\left[-\frac{\Lambda^{d-2}}{2T}\int d^dx\{(\partial_\mu\sigma)^2+\sum_{i=1}^{n-1}(\partial_\mu\pi^i)^2+\vec{J}\cdot\vec{\pi}\}\right], \tag{1}$$

in which T is a dimensionless coupling constant (proportional to the temperature in the analogous Heisenberg problem). A regularization is meant, which should preserve the $O(n)$ symmetry. It can be obtained through the Heisenberg model on a lattice with spacing $1/\Lambda$ and nearest-neighbor interactions:

$$Z=\int\prod_i[d\sigma_i\, d^{n-1}\pi_i\,\delta(\sigma_i^2+\vec{\pi}_i^2-1)]\exp[T^{-1}\{\textstyle\sum'(\sigma_i\sigma_j+\vec{\pi}_i\cdot\vec{\pi}_j)+\sum_i\vec{J}_i\cdot\vec{\pi}_i\}], \tag{2}$$

which gives a finite meaning to the invariant measure $\prod_x \delta(\sigma^2+\pi^2-1)$, restricts the momentum integrals to the zone $-\pi\Lambda < p_\alpha < \pi\Lambda$, $\alpha=(1,\ldots,d)$, and uses as inverse propagator

$$2\Lambda^2 \sum_{\alpha=1}^{d} [1-\cos(p_\alpha/\Lambda)].$$

Conversely if one starts with the Heisenberg model in the low-temperature expansion derived from (2) by integrating first over the σ field and performing then a loop expansion, one discovers in the infrared limit a regularized form of the nonlinear σ model.

Let us perform now the power counting of the primitive divergences. Since the canonical dimension of the field is $\frac{1}{2}(d-2)$, the dimension of the $2N$-point interaction vertex, obtained by expanding $[\partial_\mu(1-\vec{\pi}^2)^{1/2}]^2$, is $N(d-2)+2$. Each finite order of the loop expansion involves only a finite number of vertices of the Lagrangian. Therefore for $d-2$ small enough, the theory can be renormalized by subtracting twice each Green's function. Thus the most general counter terms are arbitrary local functions of the π field with at most two derivatives. Furthermore the regularization preserves the chiral invariance, hence the renormalized Lagrangian possesses this invariance. These two conditions fix the renormalized Lagrangian to be

$$\mathcal{L} = \frac{\mu^{d-2}}{2Z_1T_R}\int d^dx\{Z(\partial_\pi\vec{\pi}_R)^2+[\partial_\mu(1-Z\vec{\pi}_R{}^2)^{1/2}]^2\}, \tag{3}$$

i.e., all infinities may be absorbed in a field-strength and a coupling-constant renormalization. The parameter μ fixes the scale of the renormalized theory.

Renormalization-group equations.—The vertex functions of the σ and π fields[5] satisfy the equation

$$\Gamma_R{}^{(N)}(T_R,\mu) = Z^{N/2}\Gamma^{(N)}(T,\Lambda),$$

from which follows by differentiation with respect to Λ, at fixed T_R and μ,

$$\left\{\Lambda\frac{\partial}{\partial\Lambda}+W(T)\frac{\partial}{\partial T}-\frac{N}{2}\zeta(T)\right\}\Gamma^{(N)}(T,\Lambda)=0, \tag{4}$$

with[6]

$$\zeta(T) = -\Lambda(\partial \ln Z/\partial\Lambda)|_R, \tag{5}$$

$$W(T) = \Lambda(\partial T/\partial\Lambda)|_R. \tag{6}$$

These equations are sufficient for statistical mechanics[7]; for field theory it is more convenient to write the renormalization-group equation for $\Gamma_R{}^{(N)}$:

$$\left[\mu\frac{\partial}{\partial\mu}+W(T_R)\frac{\partial}{\partial T_R}-\frac{N}{2}\zeta(T_R)\right]\Gamma_R{}^{(N)}(T_R,\mu)=0.$$

If one deals with external σ lines it is easier to calculate the connected $G^{(N)}$ rather than the vertex functions $\Gamma^{(N)}$. They satisfy

$$\left[\Lambda\frac{\partial}{\partial\Lambda}+W(T)\frac{\partial}{\partial T}+\frac{N}{2}\zeta(T)\right]G^{(N)}(T,\Lambda)=0. \tag{7}$$

It remains to calculate the coefficients $W(T)$ and $\zeta(T)$ at lowest order. A one-loop (o.l.) calculation of the spontaneous magnetization $\sigma(T)$ gives

$$\sigma(T) = \langle\sigma\rangle_{\mathrm{o.l.}} = 1-(n-1)T/4\pi(d-2), \tag{8}$$

in which we have used

$$\int_{-\pi}^{+\pi}\frac{d^dp}{(2\pi)^d}\frac{1}{\sum_1{}^d 2(1-\cos p_\alpha)} = \frac{1}{2\pi(d-2)}+O((d-2)^0). \tag{9}$$

At the same order the $\pi\pi$ vertex function is

$$\Gamma_{\pi_\alpha\pi_\beta}{}^{(2)\mathrm{o.l.}} = \delta_{\alpha\beta}\frac{\Lambda^{d-2}}{T}p^2\left\{1+T\left[\frac{1}{2\pi(d-2)}+O((d-2)^0)\right]\right\}. \tag{10}$$

Equations (4), (7), (8), and (10) fix the one-loop contributions to W and ζ to be

$$\zeta(T)=[(n-1)/2\pi]\,T+O(T^2,\,T(d-2)), \tag{11}$$

$$W(T)=(d-2)T-(n-2)T^2/2\pi+O(T^3,\,T^2(d-2)). \tag{12}$$

Thus for $n>2$, $d>2$ there is an uv stable fixed point

$$T_c=2\pi(d-2)/(n-2)+O((d-2)^2), \tag{13}$$

which is the critical temperature of the Heisenberg model.[8]

Scaling behavior.—The renormalization-group equations (4) and (7), together with the canonical dimension d of the $\Gamma^{(N)}$, lead to the scaling property[9]

$$\Gamma^{(N)}(p,T,\Lambda)=\xi^{-d}(T)\sigma^{-N}(T)\Phi^{(N)}(p\,\xi(T)) \tag{14}$$

in which we have used a correlation length

$$\xi(T)=\frac{1}{\Lambda}T^{1/(d-2)}\exp\int_0^T dT'\left(\frac{1}{W(T')}-\frac{1}{(d-2)T'}\right) \tag{15}$$

and a spontaneous magnetization

$$\sigma(T)=\langle\sigma\rangle=\exp[-\tfrac{1}{2}\textstyle\int_0^T\zeta(T')\,dT'/W(T')]. \tag{16}$$

Formulas (14)–(16) hold for $T<T_c$. The exponents of the Heisenberg model follow from these formulas. From (5) and (6) we find

$$\xi(T)\sim(T_c-T)^{-\nu}\text{ with }\nu^{-1}=-W'(T_c), \tag{17}$$

$$\sigma(T)\sim(T_c-T)^{\beta}\text{ with }\beta=-\zeta(T_c)/2W'(T_c). \tag{18}$$

At one-loop orders one finds

$$\sigma(T)=\left(1-\frac{T}{T_c}\right)^{(n-1)/2(n-2)},\text{ i.e., }\beta=\frac{n-1}{2(n-2)}+O(d-2); \tag{19}$$

$$\xi(T)=\xi_0(T_c/T-1)^{-1/(d-2)},\text{ i.e., }\nu^{-1}=d-2+O((d-2)^2). \tag{20}$$

Formulas (19) and (20) re-sum the leading singularities in $d-2$ to all orders in T. At next order the results are

$$\frac{1}{\nu}=d-2+\frac{(d-2)^2}{n-2}+O((d-2)^3), \tag{21}$$

$$\eta=\frac{d-2}{n-2}-\frac{(n-1)}{(n-2)^2}(d-2)^2+O((d-2)^3). \tag{22}$$

These expressions are valid for $T<T_c$ and the domain of applicability vanishes when d goes to 2. It is thus important to be able to continue the theory above T_c.

The coefficients $W(T)$ and $\zeta(T)$ of the renormalization-group equations are regular at T_c, and therefore the Green's functions have the same scaling properties above T_c. In order to calculate explicitly the scale invariant functions above T_c, it is necessary to introduce a new interaction of the σ field with an external magnetic field H, which gives a mass to the Goldstone bosons. Then the correlation functions become regular at T_c and have a finite limit when H goes to zero above T_c. This combined use of an external source together with the renormalization-group equation allows one to go into the phase in which the symmetry is not spontaneously broken. From the correspondence with the Heisenberg ferromagnet, we know that the infrared singularities lead to a "phase transition" characterized by a restored symmetry with the generation of a mass for the pion and the σ particles. This mass can be calculated from the renormalization-group arguments and is given by[10]

$$m=\xi^{-1}(T)=\Lambda\exp\textstyle\int_T dT'/W(T').$$

It seems possible to reach the limit $d=2$ by this method. This implies that if one discusses situations of particles which have masses $\ll \Lambda$, the bare coupling constant T is of order $1/\ln\Lambda$.

The renormalization-group equation for the magnetic field expressed in terms of the magnetication reads

$$\left\{\tfrac{1}{2}\zeta(T)M\frac{\partial}{\partial M}+W(T)\frac{\partial}{\partial T}+\frac{\zeta}{2}+\frac{W(T)}{T}-d\right\}H(M,T)=0. \tag{23}$$

Its solution can be written

$$\frac{H}{M^{\delta}}=T\frac{\xi^{-d}(T)}{\sigma^{1+\delta}(T)}F\left(\left(\frac{\sigma(T)}{M}\right)\right)^{1/\beta}, \tag{24}$$

which exhibits the scaling behavior near T_c together with the Goldstone singularities. A one-loop calculation yields this relation in the form

$$\left(\frac{M}{\sigma(T)}\right)^{1/\beta}=1+\left(\frac{H\sigma(T)\xi^{d}(T)}{T}\right)^{(d-2)/2}+O(H^{d-2}). \tag{25}$$

In conclusion we would like to point out that the method presented here may be systematically extended to all orders in $d-2$, giving new expansions for the exponents and the scaling functions. It gives a combined treatment of the critical and Goldstone-like singularities. It gives an extension to higher dimensions for any asymptotically free theory. It may allow one to study the infrared problem in an asymptotically free theory although perturbation theory cannot be directly applied.

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[1]A. A. Migdal, Zh. Eksp. Teor. Fiz. 69, 1457 (1975).

[2]A. M. Polyakov, Phys. Lett. 59B, 79 (1975).

[3]G. Parisi, Nota Interna Istituto di Fisica "G. Marconi," Instituto Nazionale di Fisica Nucleare—Roma (unpublished), and K. Symanzik, DESY Report No. 75/24 (unpublished), have discussed the possibility of defining a Φ^4 theory in $4+\epsilon$ dimensions.

[4]K. Wilson and J. Kogut, Phys. Rep. 12C, 75 (1974).

[5]The $\Gamma^{(N)}$ are defined here by Legendre transformation on the sources of the fields σ as well as $\vec{\pi}$.

[6]To avoid any confusion with the inverse temperature of the critical exponents, we denote by W the Callan-Symanzik β function. A discussion of the "bare" renormalization group is given in J. Zinn-Justin, unpublished.

[7]See, for instance, E. Brézin, J. C. Le Guillou, and J. Zinn-Justin, in *Phase Transitions and Critical Phenomena,* edited by C. Domb and M. Green (Academic, New York, to be published), Vol. VI.

[8]The Abelian $O(2)$-symmetric case is special, as for gauge theories. The critical temperature remains finite in two dimensions. The field-theory and the statistical-mechanics aspects of this problem have been discussed respectively by S. Coleman, Phys. Rev. D 11, 2088 (1975), and by A. Luther and D. Scalapino, unpublished.

[9]For the two-point function B. Halperin and P. C. Hohenberg, Phys. Rev. 177, 952 (1969), had obtained a similar result from hydrodynamic considerations.

[10]This phenomenon had been noted by D. Gross and A. Neveu, Phys. Rev. D 10, 3235 (1974), in their study of the large-n limit of the four-fermion interaction.