[1]G. 't Hooft, Nucl. Phys. B79, 276 (1974).

[2]J. P. Hsu, Lett. Nuovo Cimento 14, 189 (1975); J. P. Hsu and E. Mac, to be published.

[3]J. Arafune, P. G. O. Freund, and C. J. Goebel, J. Math. Phys. (N.Y.) 16, 433 (1975); M. K. Prasad and C. M. Sommerfield, Phys. Rev. Lett. 35, 760 (1975).

[4]S. Weinberg, Phys. Rev. Lett. 19, 1264 (1967). There is a change of sign for g' in the present Eqs. (1)–(5).

[5]J. P. Hsu and E. C. G. Sudarshan, Lett. Nuovo Cimento 13, 557 (1975), and Phys. Lett. 51B, 349 (1975), and Nucl. Phys. B91, 477 (1975).

[6]T. T. Wu and C. N. Yang, in *Properties of Matter under Unusual Conditions*, edited by H. Mark and S. Fernbac (Interscience, New York, 1969), pp. 349–354; M. Ikeda and Y. Miyachi, Prog. Theor. Phys. 27, 474 (1962); G. Rosen, J. Math. Phys. (N.Y.) 13, 595 (1972).

[7]C. N. Yang and R. R. Mills, Phys. Rev. 96, 191 (1954).

[8]J. Schwinger, Phys. Rev. 144, 1087 (1966).

[9]P. A. M. Dirac, Proc. Roy. Soc., London, Ser. A 133, 60 (1934), and Phys. Rev. 74, 817 (1948).

[10]D. H. Perkins, in *Proceedings of the Fifth Hawaii Topical Conference on Particle Physics, Honolulu, Hawaii, 1973*, edited by P. N. Dobson, Jr., V. Z. Peterson, and S. F. Tuan (Univ. of Hawaii, Honolulu, 1974), p. 596. See also, J. Saction, in *Proceedings of the Seventeenth International Conference on High Energy Physics, London, 1974*, edited by J. R. Smith (Rutherford High Energy Laboratory, Didcot, Berkshire, England, 1975), p. IV–123. Experimental values of $\sin^2\theta$ range from ~ 0.1 to ~ 0.9.

[11]It was stressed in Ref. 7 that local gauge symmetry leads to gauge fields and completely determines their interaction dynamics. In a similar sense, the magnetic monopole is a consequence of local gauge symmetry. See also J. P. Hsu, Lett. Nuovo Cimento 11, 525 (1974), and 12, 128 (1975); J. P. Hsu and J. A. Underwood, Phys. Rev. D 12, 620 (1975).

[12]Both the unit isovector v^a and the ratio $\phi^a/|\phi|$ for Higgs fields ϕ^a (see Ref. 1) have nothing to do with dynamics at the classical level. For example, one looks for the solution ϕ^a of the form $\phi^a = r^a \xi(r)$, where $\xi(r)$ is to be determined dynamically; the ratio $\phi^a/|\phi|$ does not involve $\xi(r)$ and hence has nothing to do with dynamics.

[13]G. Rosen, J. Math. Phys. (N.Y.) 9, 996, 999 (1968), and references cited therein.

[14]This is because $e = -g\sin\theta$ and $A^\mu = A_3{}^\mu \sin\theta + B^\mu \cos\theta$ [as shown in Eqs. (5) and (4)] and, therefore, the assumption of the Dirac condition $e_m e = \frac{1}{2}$ do not lead to contradiction in Weinberg's theory. Note that in a *trivial* SU(2) $\otimes$ U(1) theory in which $A^\mu = A_3{}^\mu$ and $g = -e$, one can derive the condition $e_m e = 1$ so that it is inconsistent to assume $e_m e = \frac{1}{2}$. Also, in SU(2) theory one must have $e_m e = 1$; see Tai Tsun Wu and Chen Ning Yang, Phys. Rev. D 12, 3845 (1975).

[15]For other theoretical predictions of $\sin^2\theta$, see S. Weinberg, Phys. Rev. D 5, 1962 (1972); R. Acharya and Z. Horvath, Lett. Nuovo Cimento 6, 464 (1973); H. Georgi, H. R. Quinn, and S. Weinberg, Phys. Rev. Lett. 33, 451 (1974).

Magnetic Confinement in Non-Abelian-Gauge Field Theory

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A confined magnetic flux solution of finite length and finite energy, arising from non-Abelian-gauge theory, is presented.

An interesting possibility for a quark confinement mechanism which gives rise to a hadronic stringlike structure has been proposed by Nielsen and Olesen[1] and further developed by Nambu.[2] The first-named authors rediscovered the quantum flux line which threads its way through a superconductor, identifying it with the dual string. A mechanism of flux-line termination through use of Dirac monopoles[3] was then proposed by Nambu. The important point is the compatibility of the magnetic monopole and flux-line quantization. The former radiates a total amount of flux which can be exactly channeled into the flux line. The ends of the string are then a monopole-antimonopole pair, presumably a model for mesons. Baryons are yet to be constructed.

The present work is designed to produce the Nambu confinement scheme through classical *finite*-energy solutions of non-Abelian-gauge field theory. No recourse is made to Dirac monopoles. Instead we have used 't Hooft's result, namely that monopoles occur as fundamental solutions in non-Abelian-gauge field theory[4] [in his example SU(2)]. In this first step there are two massive and one massless gauge fields, the latter precluding the existence of flux lines. How to get flux lines in non-Abelian theory is shown in the Appendix of Ref. 1 where two (or more) isovector scalar mesons are introduced having noncollinear expectation values in isospin space. Such a theory has three massive gauge vector mesons. The question is then posed: Is such a field theory possessed of monopoles as well and if so, can the Nambu confinement scheme occur? In this paper we produce the affirmative answer.

In order to communicate our result in understandable and physical terms we have deemed it essential to review the various building blocks which have been incorporated into our construction (and thinking). We shall therefore present first Nambu's and 't Hooft's results in a manner cogent to our purpose and then the multiscalar model leading to magnetic flux confinement.

(1) *The Nambu model.*—Here as throughout, we shall look for static solutions. A Dirac monopole at the point a can be introduced in the gauge $\nabla\cdot\vec{A}=0$ by the following singular solution[5] of $\Delta\vec{A}=0$:

$$\begin{aligned}\vec{A}_m &= (\Phi/4\pi)\int^a d\vec{s}\,'\times\nabla(-1/|\vec{r}-\vec{r}\,'|),\\ \vec{B}_m &\equiv \nabla\times\vec{A}_m = (\Phi/4\pi)(\vec{r}-\vec{r}_a)/|\vec{r}-\vec{r}_a|^3,\end{aligned}\tag{1}$$

where the integral is taken along an arbitrary semi-infinite line (Dirac string). Nonobservability of the string in quantum mechanics requires the Dirac quantization condition $\Phi=2\pi n/e$.

In the presence of a superconductor, the current $\vec{J}=-ie[\chi^*\nabla\chi-(\nabla\chi^*)\chi]+2e^2|\chi|^2\vec{A}$ is the source of $\vec{A}$; χ is the complex order parameter $|\chi|\times\exp(-i\omega)$ which can be viewed as a complex scalar field. Far from an electromagnetic disturbance, $|\chi|$ tends to a constant value, $|\chi_C|$, on the scale of μ_s^{-1} (μ_s is the scalar mass). In this paper we take $\mu_v\ll\mu_s$ when μ_v is the vector mass ($\mu_v^2=2e^2|\chi_C|^2$). We refer to the "London approximation" as the neglect of $O(\mu_v/\mu_s)$; in this approximation $\vec{J}=\mu_v^2[-\nabla\omega/e+\vec{A}]$. A combined monopole–flux line can then be constructed explicitly from a singular solution to $\Delta\vec{A}=\vec{J}$; the result in the gauge where χ is real is[2]

$$\vec{A}=\frac{\Phi}{4\pi}\int^a d\vec{s}\,'\times\nabla\left[-\frac{\exp(-\mu_v|\vec{r}-\vec{r}\,'|)}{|\vec{r}-\vec{r}\,'|}\right],\tag{2}$$

where the integral is taken along the string singularity. For a monopole-antimonopole pair, the lower limit in (2) is taken at the position of the antipole, b. From (2) one calculates $\vec{B}$ to be

$$\vec{B}=(\Phi/4\pi)\{-\mu_v^2\int_b^a d\vec{s}\,'\exp(-\mu_v|\vec{r}-\vec{r}|)/|\vec{r}-\vec{r}\,'|+\nabla[-\exp(-\mu_v|\vec{r}-\vec{r}_a|)/|\vec{r}-\vec{r}_a|]-\nabla[-\exp(-\mu_v|\vec{r}-\vec{r}_b|)/|\vec{r}-\vec{r}_b|]\},\tag{3}$$

and

$$\nabla\cdot\vec{B}=\Phi[\delta^3(\vec{r}-\vec{r}_a)-\delta^3(\vec{r}-\vec{r}_b)].$$

Around each pole the flux emerges or recedes spherically up to a radial distance of μ_v^{-1} whereupon it becomes canalized into a flux line of diameter μ_v^{-1} which flows between the poles.

It is instructive at this point to dwell upon the gauge properties of (2) and (3). One may write $\vec{A}=\vec{A}_{\rm reg}+\vec{A}_m$ where from (1) and (2) we have $\vec{A}_{\rm reg}=(\Phi/4\pi)\int^a d\vec{s}\,'\times\nabla\{[1-\exp(-\mu_v|\vec{r}-\vec{r}\,'|)/|\vec{r}-\vec{r}\,'|\}$. $\vec{A}_{\rm reg}$ generates a conserved flux Φ which is absorbed outside the flux line by the poles due to $\vec{A}_m$. Thus we may replace $\vec{A}_m$ by $\vec{A}_m{}'$ evaluated along another Dirac string. It is easily shown that the difference $e[\vec{A}_m{}'-\vec{A}_m]$ is equal to the gradient of a function which then can be identified with ω, the phase of χ.

(2) *The 't Hooft monopole.*—In the presence of an isovector scalar field the SU(2) gauge static equations are (k, l, and m, space indices; a, b, and c, isospin indices; $\vec{v}$ denotes a space vector, $\underline{v}$ an isovector)

$$\underline{D}_k\underline{G}_{kl}=\underline{J}_l=g\underline{\psi}\times\underline{D}_l\underline{\psi},\quad \underline{D}_k\underline{D}_k\underline{\psi}=\partial V/\partial\underline{\psi},\tag{4}$$

where for any isovector $\underline{v}$, $\underline{D}_k\underline{v}\equiv\partial_k\underline{v}-g\underline{A}_k\times\underline{v}$ and

$$\underline{G}_{kl}=\partial_l\underline{A}_k-\partial_k\underline{A}_l+g\underline{A}_k\times\underline{A}_l;$$

V is a potential. Following Arafune, Freund, and Goebel,[6] a convenient method for finding monopole solutions is obtained by working in the "Abelian gauge," the latter defined by $\underline{\psi}$ being unidirectional in isospace, say the c direction. One sees from (4) that a singular solution describing a monopole at the origin is then obtained by setting $\psi_c = \text{constant} \equiv F$, the massive components $\vec{A}_a = \vec{A}_b = 0$, and $\vec{A}_c = \vec{A}_m$ as given by (1) where we take the string along the negative z axis. Perform now a gauge transformation to the "spherical gauge" $\underline{\psi}' = \Omega\underline{\psi}$; $\underline{t}\cdot\vec{\underline{A}}' = \underline{t}\cdot\vec{\underline{A}} + (i/g)\Omega^{-1}\nabla\Omega$; $\Omega = \exp(-i\varphi t_3)\exp(-i\theta t_2)\exp(i\varphi t_3)$. One obtains $\underline{\psi}' = (\underline{r}/r)F$ with $\underline{r}$ identified to $\vec{r}$, $A_{ak}' = \epsilon_{kab}r_b/gr^2$ *provided* $\Phi = 4\pi/g$. This is 't Hooft's quantization rule; in this case the string singularity has been eliminated by cancelation against the gauge term $(i/g)\Omega^{-1}\nabla\Omega$ and one is left only with a singularity at $\vec{r} = 0$. A regular solution displaying finite energy may now be constructed by introducing form factors $a(r)$ and $f(r)$, namely, $\underline{\psi}' = (\underline{r}/r)Ff(r)$ and $A_{ak}' = \epsilon_{kab}(r_b/gr^2) \times a(r)$; these are determined from (4) and the boundary conditions

$$\lim_{r\to\infty} f(r) = a(r) = 1$$

and $f(0) = a(0) = 0$. The latter condition is required for the finiteness of the energy. The solution in an idealized limit[7] is $a(r) = 1 - \mu r/\sinh(\mu r)$, $f(r) = \coth(\mu r) - 1/\mu r$ ($\mu = gF$). Thus the regular solution departs from the singular one within μ^{-1} of the origin. The crucial point is that the introduction of the form factors has not modified the pointlike character of the monopole. This is easily seen by writing the gauge-invariant magnetic field in the form[6]

$$\underline{F}_{kl} = \partial_l \underline{D}_k - \partial_k \underline{D}_l - g^{-1}\hat{\underline{\psi}}\partial_k\hat{\underline{\psi}}\times\partial_l\hat{\underline{\psi}}; \qquad \vec{D} = \hat{\underline{\psi}}\cdot\vec{\underline{A}}; \quad \hat{\underline{\psi}} = \underline{\psi}/|\underline{\psi}|. \tag{5}$$

In the spherical gauge the magnetic monopole field entirely originates from the second term and is independent of $f(r)$ and $a(r)$.

A dipole can be similarly constructed in this model with the $\vec{B}$ lines of force of a conventional dipole and hence "unconfined" [the relevant gauge transformation[6] is now $\Omega' = \exp(-i\varphi t_3) \times \exp(-i\delta t_2)\exp(i\varphi t_3)$, where δ is the angle from which the dipole is seen]. In order to have a confined solution the isospin component of $\underline{A}$ parallel to $\underline{\psi}$ must pick up a mass. This is achieved as follows.

(3) *The multiscalar model.*—Following the appendix of Ref. 1 we introduce more than one isovector scalar. A convenient choice is to work with three $\underline{\varphi}_i$ ($i = 1,2,3$) wherein $V(\underline{\varphi}_i)$ is permutation symmetric. The Lagrangian is (static case)

$$L = -\tfrac{1}{2}(\underline{D}_k\underline{\varphi}_i)\cdot(\underline{D}_k\underline{\varphi}_i) - \tfrac{1}{4}\underline{G}_{kl}\cdot\underline{G}_{kl} - V(\underline{\varphi}_i). \tag{6}$$

The natural fields with which to work are the linear combinations which form bases of the irreducible representations of the permutation group. These are $\underline{\psi} = (1/\sqrt{3})(\underline{\varphi}_1 + \underline{\varphi}_2 + \underline{\varphi}_3)$ [singlet] and the complex doublet $\underline{\chi} = (1/\sqrt{3})[\underline{\varphi}_1 + \exp(2\pi i/3)\underline{\varphi}_2 + \exp(4\pi i/3)\underline{\varphi}_3]$. The detailed choice of V is without importance, the relevant point being that its minimum leads to a symmetric pyramidal arrangement as depicted in Fig. 1 where the symmetry axis is chosen to be c. The minima are then $\psi_a = \psi_b = 0$, $\psi \equiv \psi_c = \sqrt{3}F\cos\alpha$; $\chi_c = 0$, $\chi^+ = (1/\sqrt{2}) \times (\chi_a + i\chi_b) = 0$, $\chi \equiv \chi^- = (1/\sqrt{2})(\chi_a - i\chi_b) = \sqrt{\tfrac{3}{2}}F\sin\alpha \times \exp(-i\omega)$. For example,

$$V = -\tfrac{1}{2}\mu_0^2(\underline{\psi}\cdot\underline{\psi} + 2\underline{\chi}^*\cdot\underline{\chi}) + \tfrac{1}{4}\lambda(\underline{\psi}\cdot\underline{\psi})^2 + \lambda'(\underline{\chi}^*\cdot\underline{\chi})^2. \tag{7}$$

Then $\tan^2\alpha = \lambda/\lambda'$, a parameter ranging from 0 to ∞.

The equations for $\vec{\underline{A}}$ are of the form (4) where $\underline{J}_l = g[\underline{\psi}\times\underline{D}_l\underline{\psi} + \underline{\chi}\times\underline{D}_l\underline{\chi}^* + \underline{\chi}^*\times\underline{D}_l\underline{\chi}]$. The vector meson mass matrix arises from the linear term in $\vec{\underline{A}}$ in $\vec{\underline{J}}$. One readily deduces that its eigenvectors are in the transverse a-b plane (twofold degenerate) and in the longitudinal c direction with masses given by $\mu_T^2 = 3F^2g^2(1 - \sin^2\alpha/2)$; $\mu_L^2 = 3F^2g^2\sin^2\alpha$, respectively.

We now construct a monopole–flux-line solution by letting the fields vary in space keeping the pyramidal symmetry. First select an "Abelian" gauge where $\underline{\psi}$ points everywhere in the c direc-

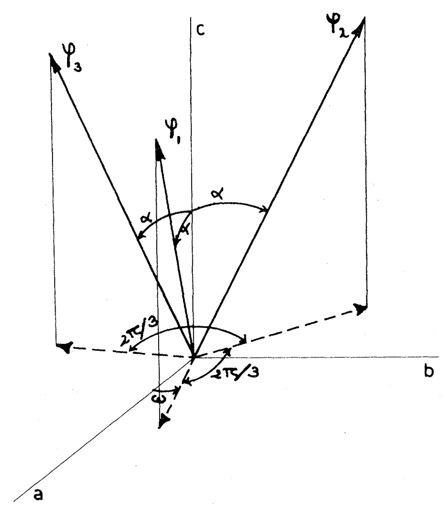

FIG. 1. Pyramidal symmetry of the fields at the potential minimum $|\underline{\varphi}_1| = |\underline{\varphi}_2| = |\underline{\varphi}_3| = F$.

tion. It is easily checked once more that the equations of motion of G_{akl} and G_{bkl} contain terms having at least one factor of A_{ak} and/or A_{bk}; moreover A_{ck} does not enter the equation for $\underline{\psi}$. Thus we look for a singular solution by setting first $\vec{A}_a = \vec{A}_b = 0$; $\psi \equiv \psi_c = \text{constant} = \sqrt{3}F(\vec{r}) \cos\alpha(\vec{r})$. The remaining equations are [from (6) and (7)]

$$\Delta \vec{A}_c = -ig[\chi^* \nabla \chi - (\nabla \chi^*)\chi] + 2g^2 |\chi|^2 \vec{A}_c, \tag{8}$$

$$(\nabla + ig\vec{A}_c)(\nabla + ig\vec{A}_c)\chi = \frac{\partial V}{\partial |\chi|^2}\chi. \tag{9}$$

Equations (8) and (9) are our central result since they permit the construction of a singular solution to (6) which has *all* the properties of the Nambu model. In fact in the London approximation, the result (2) is recovered for $\vec{A}_c$ with the identification $\mu_v = \mu_L$, in a gauge where χ is real. The exact singular solution to (8) and (9) exhibiting a monopole of order n differs from the London approximation in two important respects. (a) The regular part of $\vec{A}_c$ in the decomposition $A_c = \vec{A}_{c,\text{reg}} + \vec{A}_m$ [$\vec{A}_m$ given by (1)] now leads to a finite $\vec{B}$ field along the string instead of the logarithmically divergent result (3); the cutoff is provided by the mass m_χ of the χ field.[2] (b) Because the string singularity remains in the exact solution, χ vanishes along the string (real gauge) as seen from (9). This result implies that in any Abelian gauge $|\chi|$ vanishes along the center of the flux line. In fact, taking a flux line centered along the negative z axis, one can solve (9) in the vicinity of the origin. One finds $|\chi| \sim \exp\{\frac{1}{2}[(1+2|n|)^{1/2}-1]\ln r\}(1+\cos\theta)^{|n|/2}$, $r \ll \mu_L^{-1}$. For $z<0$, $r \gg \mu_L^{-1}$, and $\rho \ll \mu_L^{-1}$ where $\rho = (r^2 - z^2)^{1/2}$, one can again solve (9) to recover the familiar result[8] $|\chi| \sim \rho^{|n|}$.

We now perform the gauge transformation Ω to the spherical gauge and this reduces again the line singularity to a point singularity for $n=2$, corresponding to $\Phi = 4\pi/g$ or two units of quantum flux. The $\vec{B}$ field due to $\vec{A}_m$ is carried by the scalar term in (5) and we obtain $\vec{A}_{c,\text{reg}} = \vec{D}$. One can now construct a regular solution as above which tends to the singular one for distances $r \gg \mu_T^{-1}$. This still does not affect the monopole fields and modifies the $\vec{D}$ field in (5) in a smooth way. Once more the infinite energy located at the pole is removed. If $\tan^2\alpha \ll 1$ the picture given by Eq. (2) and (3) will remain qualitatively correct except for the logarithmic cutoff mentioned above. In addition, the shape of the form factor near the monopole will be unaffected. If this condition is not met the $\vec{B}$ field may be distorted but this will have no effect in the region of the well-formed flux line. At all events the differential equations for the form factors are continuous in α, therefore guaranteeing the existence of a solution with finite energy density.

Thus the 't Hooft monopole of finite energy now becomes an infinite-energy phenomenon (since the accompanying flux line is of infinite length). However the dipole solution can be constructed in a straightforward way using the gauge transformation Ω' to remove the string singularity. The result is thus a confined solution with finite energy. For $l \gg \mu_L^{-1} \gg \mu_T^{-1}$ $(\alpha \ll 1)$, where l is the distance between the two monopoles, this energy is roughly the sum of two terms: (a) monopole mass terms of order[4] μ_T/g and (b) a term proportional to l arising from the flux line.[2] We do not discuss the stability of such a solution; this would require taking into account the quantized motion of the monopoles. Note however that in the strong-coupling limit $g \gg 1$, the small monopole mass should prevent pair annihilation into vector and scalar mesons.

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[1]H. B. Nielsen and P. Olesen, Nucl. Phys. B61, 45 (1973).

[2]Y. Nambu, Phys. Rev. D 10, 4262 (1974). See also G. Parisi, Phys. Rev. D 11, 970 (1975); M. Creutz, Phys. Rev. D. 10, 2696 (1974).

[3]P. A. M. Dirac, Phys. Rev. 74, 817 (1948).

[4]G. 't Hooft, Nucl. Phys. B79, 276 (1974).

[5]By singular solution we shall always mean a solution almost everywhere; it may fail on lines or points. In Eqs. (1) and (2), $\vec{A}$ is not defined on the string. We specify that the only singularity of $\vec{A}$ is the infinity arising when approaching the string from outside. This distinguishes the string from an idealized solenoid where $\vec{A}$ is put equal to zero on this line; the resulting infinite slope gives rise to a flux Φ flowing along the line to the extremity a.

[6]S. Arafune, P. G. O. Freund, and C. S. Goebel, J. Math. Phys. (N. Y.) 16, 433 (1975).

[7]M. K. Prasad and C. M. Sommerfield, Phys. Rev. Lett. 35, 760 (1975).

[8]See for instance A. L. Fetter and P. C. Hohenberg, in *Superconductivity*, edited by R. D. Parks (Marcel Dekker, New York, 1969), Vol. II, p. 817.