Mixing Angle θ and Magnetic Monopole in Weinberg's Unified Gauge Theory*

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(Received 6 August 1975)

Gauge symmetry admits a local unit isovector and leads to the magnetic monopoles in Weinberg's unified theory. I predict $\sin^2\theta = \frac{1}{2}$ for the mixing angle θ on the basis of Dirac's condition for charge quantization. This interesting result should be tested experimentally.

The magnetic monopole has been shown to exist in certain non-Abelian gauge theories, excluding Weinberg's unified theory.[1,2] The magnetic charge follows from the topological structure of three Higgs scalar fields in a three-dimensional space.[3]

Here, I wish to point out that one can always introduce a local unit isovector in a non-Abelian gauge theory to substitute for the role played by the three Higgs scalars, so far as the magnetic monopole is concerned. Thus, the requirement that the electromagnetic group be a subgroup of a larger group with a compact covering, as in Ref. 1, is not necessary for the monopole solution.[2] In such a formulation of the monopoles, the magnetic charge and its conservation have nothing to do with the topology of Higgs fields and the dynamics of gauge fields. They are simply consequences of the local isospin gauge symmetry. I get exact solutions for the vector gauge fields and show the presence of a stable monopole with the magnetic charge $e_m = -\sin\theta/g$ in Weinberg's unified theory. On the basis of the Dirac condition for charge quantization, the theory predicts $\sin^2\theta = \frac{1}{2}$ for the mixing angle θ, which can be tested experimentally.

In Weinberg's theory, the equations for the classical fields $\vec{A}_\mu$, B_μ, φ, and $\varphi^\dagger$ are[4,5]

$$\partial_\mu \vec{A}^{\mu\nu} - g\vec{A}^{\mu\nu}\times\vec{A}_\mu + ig\Phi^{\dagger\nu}\vec{t}\varphi - ig\varphi^\dagger\vec{t}\Phi^\nu = 0, \tag{1}$$

$$\partial_\mu B^{\mu\nu} - \tfrac{1}{2}ig'\varphi^\dagger\Phi^\nu + \tfrac{1}{2}ig'\Phi^{\dagger\nu}\varphi = 0, \tag{2}$$

$$\partial_\mu\Phi^\mu - M_1^2\varphi + 2h\varphi^\dagger\varphi\varphi - (ig\vec{A}_\mu\cdot\vec{t} + \tfrac{1}{2}ig'B_\mu)\Phi^\mu = 0, \tag{3}$$

$$\varphi = \begin{pmatrix} \varphi^+ \\ (\varphi_1^0 + \sqrt{2}\lambda + i\varphi_2^0)/\sqrt{2} \end{pmatrix},$$

$$\vec{A}_{\mu\nu} \equiv \partial_\mu\vec{A}_\nu - \partial_\nu\vec{A}_\mu + g\vec{A}_\mu\times\vec{A}_\nu, \quad B_{\mu\nu} \equiv \partial_\mu B_\nu - \partial_\nu B_\mu,$$

$$\Phi^\mu \equiv \partial^\mu\varphi - ig\vec{A}^\mu\cdot\vec{t}\varphi - \tfrac{1}{2}ig'B^\mu\varphi,$$

where the leptons have been neglected for simplicity. The photon field A_μ and the neutral massive vector field Z_μ are given by

$$A^\mu = A_3^\mu\sin\theta + B^\mu\cos\theta, \quad Z^\mu = A_3^\mu\cos\theta - B^\mu\sin\theta. \tag{4}$$

The mixing angle θ and the charge e are given by

$$\tan\theta = g'/g \text{ and } e = -g\sin\theta. \tag{5}$$

We are interested in the nontrivial solutions for the vector fields A_a^μ and B^μ. The scalar fields have obviously the following trivial solutions:

$$\varphi^\pm = 0, \quad \varphi_2^0 = 0, \quad \varphi_1^0 = -\sqrt{2}\lambda = -2\sqrt{2}M_W/g, \tag{6}$$

where M_W is the mass of $W^{\pm\mu} = (A_1^\mu \mp iA_2^\mu)/2^{1/2}$. We look for the static spherically symmetric solution of the form[6]

$$A_0^a = v^a A_0(r), \quad A_i^a = \epsilon_{iab}v^b A(r), \quad v^b = r^b/r, \quad i,a,b = 1,2,3, \tag{7}$$

$$B^0 = 0, \quad B^i = v^i B(r), \quad i = 1,2,3, \tag{8}$$

where v^b is a local unit isovector. Equations (2) and (3) are satisfied by the solutions (6) and (8) with arbitrary $B(r)$. Equation (1) reduces to

$$r^2 d^2A/dr^2 + 2r\,dA/dr - A(1+grA)(2+grA) + grA_0^2(1+grA) = 0,$$
$$r^2 d^2A_0/dr^2 + 2r\,dA_0/dr - 2A_0(1+grA)^2 = 0. \tag{9}$$

The special "particlelike" solution to (9) is

$$A(r) = F/gr, \quad A_0 = 0, \quad F = -1, -2, \tag{10}$$

which has singularities of the Coulomb form. We also have the following singularity-free solution for an SU(2) gauge field,

$$A(r) = (R - \sinh R)/gr \sinh R, \quad R = \beta r, \quad \beta \text{ real},$$
$$A_0(r) = i(R\cosh R - \sinh R)/gr \sinh R. \tag{10a}$$

Note that if β is complex with $\mathrm{Re}\beta \neq 0$, then (10a) is also a solution.

To understand the meaning of the classical solutions, we define a generalized electromagnetic field tensor $\overline{F}_{\mu\nu}$ with the help of a local unit isovector $v^a(x_\mu)$:

$$\overline{F}_{\mu\nu} = v^a A_{\mu\nu}{}^a \sin\theta + B_{\mu\nu}\cos\theta - (\sin\theta/g)\epsilon^{abc}v^a(D_\mu v^b)D_\nu v^c, \tag{11}$$
$$D_\mu v^b = \partial_\mu v^b + g\epsilon^{bce}A_\mu{}^c v^e, \quad v^a(x_\mu)v^a(x_\mu) = 1.$$

As usual, the definition (11) is invariant under SU(2)$\otimes$U(1) gauge transformation and $\overline{F}_{\mu\nu}$ becomes the usual electromagnetic field tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $A^\mu = A_3{}^\mu \sin\theta + B^\mu\cos\theta$, when $v^a = (0, 0, 1)$.[1] Note that the unit isovector v^a in (11) can be a function of space-time in general because of the local isospin gauge symmetry.[7] Since $v^a v^a = 1$, we can rewrite (11) as

$$\overline{F}_{\mu\nu} = [\partial_\mu(v^a A_\nu{}^a) - \partial_\nu(v^a A_\mu{}^a)]\sin\theta + B_{\mu\nu}\cos\theta - (\sin\theta/g)\epsilon^{abc}v^a\,\partial_\mu v^b\,\partial_\nu v^c. \tag{12}$$

The electric and the magnetic fields, E_j and H_k, are given by

$$H_k = \tfrac{1}{2}\epsilon_{kij}\overline{F}_{ij}, \quad E_j = \overline{F}_{j0}. \tag{13}$$

It follows from (7), (8), (10), (12), and (13) that

$$\vec{\mathrm{H}} = -\vec{\mathrm{r}}\sin\theta/gr^3, \quad \vec{\mathrm{E}} = 0. \tag{14}$$

The total magnetic flux is $-4\pi\sin\theta/g$. Thus there is a stable magnetic monopole at $\vec{\mathrm{r}} = 0$ with the magnetic charge

$$e_m = -\sin\theta/g. \tag{15}$$

From (5) and (15) we obtain

$$ee_m = \sin^2\theta. \tag{16}$$

The Schwinger condition[8] $ee_m = 1$ and (16) give the result $\cos\theta = g = 0$ and, therefore, it is incompatible with the theory because one must have $g \neq 0$ and $g' \neq 0$. The only charge-quantization condition compatible with (16) is the Dirac condition[9] $ee_m = \frac{1}{2}$, which leads to the interesting result

$$\sin^2\theta = \tfrac{1}{2}. \tag{17}$$

This implies a universal coupling, $g = g'$, for the vector fields $A_a{}^\mu$ and B^μ in Weinberg's theory. Moreover, (17) leads to $M_W{}^2 = M_Z{}^2/2 = e^2/2\sqrt{2}G_W$ and the total effective e-ν interaction $(G_W/\sqrt{2})\overline{\nu}\gamma_\mu(1+\gamma_5)\nu\overline{e}\gamma^\mu(\frac{1}{2}+\frac{3}{2}\gamma_5)e$. Thus, arbitrary features in Weinberg's theory are largely removed. The prediction (17) is consistent with the average value of various experimental results.[10] The prediction should be further tested. I stress that these unambiguous predictions in Weinberg's theory are made on the basis of simplicity and beauty in equations derived from the concepts of local gauge symmetry and charge quantization. In view of the present technical difficulty[10] in testing (17), one should not allow oneself to be too discouraged simply because there is not complete agreement between (17) and some experiments, e.g., the reactor experiment $\overline{\nu}_e + e \to e + \overline{\nu}_e$.[10]

The magnetic current $j_\lambda{}^m$ and the electric current $j_\lambda{}^e$ are related to $\overline{F}_{\mu\nu}$ by

$$j_\lambda{}^m = \tfrac{1}{2}\epsilon_{\lambda\rho\mu\nu}\partial^\rho \overline{F}^{\mu\nu}, \tag{18}$$

and

$$j_\lambda{}^e = \partial^\rho \overline{F}_{\lambda\rho}, \tag{19}$$

which are obviously conserved: $\partial^\lambda j_\lambda{}^m = \partial^\lambda j_\lambda{}^e = 0$. When the vector fields $A_a{}^\mu$ and B^μ are free from line singularity, we have[3]

$$\epsilon^{\lambda\rho\mu}\partial_\rho[\partial_\mu(v^a A_\nu{}^a) - \partial_\nu(v^a A_\mu{}^a) + \partial_\mu B_\nu - \partial_\nu B_\mu] = 0, \tag{20}$$

and the magnetic current $j_\lambda{}^m$ takes the form

$$j_\lambda{}^m = -(\sin\theta/2g)\epsilon^{abc}\epsilon_{\lambda\rho\mu\nu}(\partial^\rho v^a)(\partial^\mu v^b)\partial^\nu v^c. \tag{21}$$

It can be seen that the magnetic charge $(1/4\pi)\int j_0{}^m d^3r$ and its conservation are direct consequences of the local isospin gauge symmetry[7,11] which admits the local unit isovector $v^a(x_\lambda)$ in the theory.[12] In general, (21) with a time-independent $v^a(\vec{r})$ implies that the magnetic charge must be an integer in units of $-\sin\theta/g$.[3] The general Dirac condition $ee_m = n/2$ is satisfied if and only if (17) holds.

The value of the magnetic monopole mass M_m is of course very important for experiment. The solutions (6), (8), and (10) lead to the energy E or the monopole mass

$$M_m = E = -\int \mathcal{L}\, d^3r = \begin{cases} 0, & F=-2, \quad (22\text{a}) \\ \infty, & F=-1, \quad (22\text{b}) \end{cases}$$

for the static monopole system. The physical monopole probably could have a nonzero mass as a result of quantum corrections to (22a). The singularity-free solution (10a) does not have a simple physical interpretation because A_0 is imaginary. Yet the exact solution (10a) with β real is interesting for it leads to a finite energy:

$$E = \frac{4\pi}{g^2}\int_0^\infty dr\, \frac{d}{dr}\left(\frac{\beta R^2\cosh R}{\sinh^3 R} - \frac{\beta\cosh R}{\sinh R} - \frac{\beta R}{\sinh^2 R} + \frac{\beta}{R}\right) = \frac{4\pi}{g^2}|\beta|, \quad R=\beta r, \quad A_0 \neq 0, \tag{23}$$

where the arbitrary constant β has the dimension of a mass. According to the variational principle, we expect the finite-energy solution to exist even if $A_0 = 0$.[13] Let us consider the simple case where $A_0 = \beta_\mu = M_1 = h = 0$, while $M_1{}^2/h$ may not be zero. With the help of an arbitrary parameter m with the dimension of a mass, we may write E as

$$E = \frac{4\pi}{g^2} m \int_0^\infty dx\left[\left(\frac{d\overline{A}}{dx}\right)^2 + \frac{(\overline{A}^2 + 2A)^2}{2x^2} + \frac{(x\,d\overline{\varphi}/dx - \overline{\varphi})^2}{2x^2} + \frac{\overline{\varphi}^2(\overline{A}+2)^2}{4x^2}\right], \tag{24a}$$

$$= (4\pi/g^2)\, mI, \tag{24b}$$

where the dimensionless quantities x, $\overline{A}$, and $\overline{\varphi}$ are given by $x = mr$, $A_i{}^a = \epsilon^{iab}(r^b/r)m\overline{A}/xg$, and $\varphi^a = (r^a/r)m\overline{\varphi}/xg$. The quantity I is the minimum value of (24a) and can be found by computer calculations, using trial functions and adjusting their parameters. The numerical value of I is not important physically because m in (24b) is arbitrary and therefore E cannot be determined at the classical level.

To conclude, in contrast to 't Hooft's formalism[1] I have given a formalism in which the local gauge symmetry admits a local unit isovector and leads to the magnetic monopole with a finite mass in Weinberg's unified theory. In general, the properties of the monopole in SU(2) ⊗ U(1) theory are not necessarily exactly the same as those of the monopole in U(1) theory or SU(2) theory. For example, quantized monopole strength $e_m = 1/e$, derived from the SU(2) symmetry group, does not necessarily apply to Weinberg's theory.[14] The possibly existing monopole should be searched for experimentally without preconception, especially if the prediction (17) is confirmed.[15]

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*Work supported in part by the U. S. Atomic Energy Commission (now U. S. Energy Research and Development Administration).

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[11]It was stressed in Ref. 7 that local gauge symmetry leads to gauge fields and completely determines their interaction dynamics. In a similar sense, the magnetic monopole is a consequence of local gauge symmetry. See also J. P. Hsu, Lett. Nuovo Cimento 11, 525 (1974), and 12, 128 (1975); J. P. Hsu and J. A. Underwood, Phys. Rev. D 12, 620 (1975).

[12]Both the unit isovector v^a and the ratio $\phi^a/|\phi|$ for Higgs fields ϕ^a (see Ref. 1) have nothing to do with dynamics at the classical level. For example, one looks for the solution ϕ^a of the form $\phi^a = r^a \xi(r)$, where $\xi(r)$ is to be determined dynamically; the ratio $\phi^a/|\phi|$ does not involve $\xi(r)$ and hence has nothing to do with dynamics.

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[14]This is because $e = -g\sin\theta$ and $A^\mu = A_3{}^\mu \sin\theta + B^\mu \cos\theta$ [as shown in Eqs. (5) and (4)] and, therefore, the assumption of the Dirac condition $e_m e = \frac{1}{2}$ do not lead to contradiction in Weinberg's theory. Note that in a *trivial* SU(2) $\otimes$ U(1) theory in which $A^\mu = A_3{}^\mu$ and $g = -e$, one can derive the condition $e_m e = 1$ so that it is inconsistent to assume $e_m e = \frac{1}{2}$. Also, in SU(2) theory one must have $e_m e = 1$; see Tai Tsun Wu and Chen Ning Yang, Phys. Rev. D 12, 3845 (1975).

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Magnetic Confinement in Non-Abelian-Gauge Field Theory

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(Received 1 December 1975)

A confined magnetic flux solution of finite length and finite energy, arising from non-Abelian-gauge theory, is presented.

An interesting possibility for a quark confinement mechanism which gives rise to a hadronic stringlike structure has been proposed by Nielsen and Olesen[1] and further developed by Nambu.[2] The first-named authors rediscovered the quantum flux line which threads its way through a superconductor, identifying it with the dual string. A mechanism of flux-line termination through use