no experimental evidence.

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Circulation and Angular Momentum in the *A* Phase of Superfluid Helium-3*

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It is shown that the curl of the velocity field generally identified with $\vec{v}_s$ in the *A* phase of superfluid ^{3}He is entirely determined in the absence of singularities by the spatial gradients of the order-parameter symmetry axis $\hat{l}$. As a simple application of this relation it is argued that in a texture of cylindrical symmetry in a volume V, the liquid should have a nonvanishing thermal-equilibrium orbital angular momentum of order $\rho_s V\hbar/M$.

The broken symmetry associated with the ordering in superfluid ^{3}He-*A* gives rise to two new sets of hydrodynamic variables[1]: a velocity field $\vec{v}_s$ and the independent components of the gradient of the local symmetry axis $\hat{l}$ of the order parameter.[2] The development of a complete hydrodynamics based on these variables has been hampered by two related difficulties, of which only the first has received explicit attention:

(1) If $\hat{l}$ is not everywhere close to a fixed spatial direction then one cannot express $\vec{v}_s$ as the gradient of a global phase.[3] Perhaps as a result, hydrodynamic theories have been attempted only in the linear regime. Since it is likely that real samples of ^{3}He-*A* are characterized by textures in which, in the absence of aligning fields, the direction of $\hat{l}$ wanders slowly through large angles,[4] such linear hydrodynamics can be inadequate even when flow velocities are small. (2) It is often implicitly assumed that $\hat{l}$ and $\vec{v}_s$ are independent variables. We shall show that this too is only valid in linearized treatments. A theory of flow in the presence of finite spatial variations of $\hat{l}$ must take the constraint between $\vec{v}_s$ and $\hat{l}$ [Eq. (6) below] into account. The solution to a third problem[5] requires a resolution of the first two. (3) Does a specimen of ^{3}He-*A* in thermal equilibrium have a nonvanishing orbital angular momentum of order $\rho_s V\hbar/M$? From a macroscopic point of view this should indeed be the case, unless the terms in $\vec{v}_s$ in the equilibrium momentum density[3]

$$\vec{g}=\rho_s\vec{v}_s-\rho_0\hat{l}\hat{l}\cdot\vec{v}_s+C\nabla\times\hat{l}-C_0\hat{l}(\hat{l}\cdot\nabla\times\hat{l}) \tag{1}$$

necessarily give rise to terms in $\vec{L}=\int d^3r\,\vec{r}\times\vec{g}$ which almost entirely cancel the contribution from the other terms (which will, in general, be of this order[6]).

In this Letter we wish to resolve the first two problems and, to illustrate the utility of this resolution, use it to argue in support of an equilibrium angular momentum of order $\rho_s V\hbar/M$. To do this we return to the more fundamental characterization of the broken symmetry in terms of the complex order parameter[2,7]

$$\psi(\vec{r}_1,\vec{r}_2)=[\vec{\varphi}^1(\vec{r})+i\vec{\varphi}^2(\vec{r})]\cdot\vec{\rho}\chi(\rho),\quad \vec{r}=\tfrac{1}{2}(\vec{r}_1+\vec{r}_2),$$
$$\rho=\vec{r}_1-\vec{r}_2,\quad \vec{\varphi}^\alpha\cdot\vec{\varphi}^\beta=\delta_{\alpha\beta},\quad \hat{l}=\vec{\varphi}^1\times\vec{\varphi}^2, \tag{2}$$

whose degeneracy is fully characterized by a set of two orthonormal axes (and no additional phase variable, the overall phase of ψ being entirely controlled by the orientation of the axes). From this point of view the additional hydrodynamic variables are a set of gradients sufficient to specify the linear spatial variation in orientation of the axes.[8] This information is carried by a tensor field, $\underline{\Omega}$, such that $\delta r_j\Omega_{ji}$ gives the infinitesimal rotation $\delta\omega_i$ necessary to produce the axes at $\vec{r}+\delta\vec{r}$ from those at $\vec{r}$. Because ψ transforms like a two-particle wave function under Galilean transformations, it follows that $\underline{\Omega}$ obeys the transformation law

$$\Omega_{ij}{}^{1}=\Omega_{ij}+(2M/\hbar)u_i l_j, \tag{3}$$

where $\vec{u}$ is the velocity of the moving frame. This suggests the replacement of $\underline{\Omega}$ by a vector transforming like a velocity,

$$v_{si}=-(\hbar/2M)\Omega_{ij}l_j, \tag{4}$$

and a Galilean invariant matrix,

$$\hat{\Omega}_{ij}=\Omega_{ij}-\Omega_{ik}l_k l_j. \tag{5}$$

It follows from the basic definition of $\hat{\underline{\Omega}}$ that a knowledge of $\hat{\underline{\Omega}}$ and $\vec{l}$ at a point $\vec{r}$ is equivalent to a knowledge of $\vec{l}$ and its gradient at that point, which brings us back to the conventional variables described in the opening paragraph.

However these variables are now constrained in the absence of singularities by the condition $\nabla_i(\nabla_j\vec{\varphi}^{\alpha})=\nabla_j(\nabla_i\vec{\varphi}^{\alpha})$. The consequences of this constraint are fully accounted for by eliminating $\hat{\underline{\Omega}}$ in favor of $\vec{l}$ and its gradients *and* by noting that it requires $\vec{v}_s$ and $\vec{l}$ to be related by[9]

$$\nabla_i v_{sj}-\nabla_j v_{si}=(\hbar/2M)\vec{l}\cdot(\nabla_i\vec{l}\times\nabla_j\vec{l}). \tag{6}$$

Thus even in the absence of singularities, the curl of $\vec{v}_s$ does not vanish to second order in the deviations from uniform equilibrium, but is entirely determined by $\vec{l}$ and its gradients.

The constraint (6) is essential to a macroscopic determination of the equilibrium angular momentum in a given geometry, for it must be invoked in minimizing the free energy[10]

$$f=\int d^3r\{\tfrac{1}{2}\rho_s v_s^2-\tfrac{1}{2}\rho_0(\vec{l}\cdot\vec{v}_s)^2+C(\vec{v}_s\cdot\nabla\times\vec{l})-C_0(\vec{v}_s\cdot\vec{l})(\vec{l}\cdot\nabla\times\vec{l})+\tfrac{1}{2}K_1(\nabla\cdot\vec{l})^2+\tfrac{1}{2}K_2(\vec{l}\cdot\nabla\times\vec{l})^2+\tfrac{1}{2}K_3[\vec{l}\times(\nabla\times\vec{l})]^2\} \tag{7}$$

to determine the forms of $\vec{v}_s$ and $\vec{l}$ appearing in the momentum density (1). Details of this problem will be examined elsewhere, and we only mention here an especially simple type of stationary configuration for which many features of the total angular momentum can be deduced entirely from the constraint (6).

Consider a long cylinder of ^{3}He-A, subject to the boundary condition[11] that $\vec{l}$ be perpendicular to the surface at $r=R$, in which the equilibrium texture has full cylindrical symmetry[12]:

$$\vec{l}(r,\varphi,z)=\hat{z}l_z(r)+\hat{r}l_r(r),\quad l_z^2+l_r^2=1,\quad l_z(0)=l_r(R)=1. \tag{8}$$

The constraint (6) is satisfied by a (nonsingular) $\vec{v}_s$ of the form[13,14]

$$\vec{v}_s=(\hbar/2Mr)[1-l_z(r)]\hat{\varphi}. \tag{9}$$

The angular momentum is then

$$\vec{L}=(\hbar/2M)\rho_s\hat{z}\int d^3r\{1+[(4Mc/\hbar\rho_s)-1]l_z(r)\}, \tag{10}$$

which will be of order $\hbar\rho_s V/M$ unless some further identity requires a near cancelation of terms. The following observations bear on the possibility of such a cancelation and the general plausibility of an angular momentum of this order:

(a) We find that (8) and (9) do indeed make f stationary over all $\vec{l}$ and $\vec{v}_s$ constrained by (6), provided $l_z(r)=\cos\theta(r)$ is given by

$$\frac{r}{R}=\exp\left[-\int_{\theta(r)}^{\pi/2}\left(\frac{K_1\cos^2\theta+K_3\sin^2\theta}{K_1\sin^2\theta+K_4(1-\cos\theta)^2}\right)^{1/2}d\theta\right], \tag{11}$$

where $K_4=\rho_s(\hbar/2M)^2$. Since different parameters appear in (11) and (10), only a numerical accident can yield a cancelation.[15]

(b) One must, however, be wary of higher-order instabilities. For example, the appearance of a nonzero circumferential component of $\vec{l}$ should be considered. This, however, would entail nonvanishing axial and radial contributions to $\vec{g}$ from the terms explicitly parallel to $\vec{l}$ in (1); $\vec{v}_s$ would then have to be augmented by an irrotational part which canceled the radial current density and the net axial current, at a cost in free energy that would balance against the terms favoring alignment of $\vec{l}$ and $\vec{v}_s$. This problem is currently under study and we mention it here only to emphasize that the tendency of $\vec{l}$ to be aligned by flow[16] need not necessarily entail a circumferential collapse.

(c) Even if l_z and l_r depend on z as well as r, and the containing vessel has z-dependent top and bottom surfaces, a $\vec{v}_s$ of the form (9) continues to satisfy the constraint (6), and for such a $\vec{v}_s$ the total angular momentum continues to be given by (10). One can therefore consider shapes (such as a rather flat lens) in which the surface helps to stabilize a nonsingular texture for which L can readily be shown to be of order $\rho_s V\hbar/M$.

(d) It should also be emphasized that although an equilibrium angular momentum of order $\rho_s V\hbar/M$ is large enough to be observed and larger by factors of a thousand or a million than other predictions, it is, from the point of view of rigid-body rotations, a very small angular momentum, being comparable to that of a single quantized vortex line in ^{4}He II. Indeed, the inability of the ^{3}He-A to dispose of it by suitable counterflow is precisely due to the quantization of circulation in ^{3}He-A, which assumes a form quite similar to that in ^{4}He II except for the "zero-point circulation" required by the texture in $\vec{l}$.

We have benefited greatly from many conversations with Vinay Ambegaokar, and we are indebted to M. E. Fisher for lending us what seems to be the only copy of de Gennes's book now in Ithaca. One of us (N.D.M.) has also been influenced by S. Teukolsky's beautiful lectures on general relativity.

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[1]P. G. de Gennes, in *Proceedings of the Twenty-Fourth Nobel Symposium on Collective Properties of Physical Systems, Aspenaasgarden, Sweden, 1973*, edited by B. Lundqvist and S. Lundqvist (Academic, New York, 1974), p. 112.

[2]For the conceptual point we wish to make, the additional hydrodynamic variable associated with the d axis in spin space is an irrelevant complication, since its rotations do not couple to the overall phase of the order parameter. We therefore ignore it here, though it must be taken into account in an accurate determination of textures and can substantially alter the textural structure in magnetic fields.

[3]See, for example, V. Ambegaokar, P. G. de Gennes, and D. Rainer, Phys. Rev. A 9, 2676 (1974), and 12, 345 (1975); R. Graham, Phys. Rev. Lett. 33, 1431 (1974); M. C. Cross, J. Low Temp. Phys. 21, 525 (1975); P. Wölfle, Phys. Lett. 47A, 224 (1974).

[4]See, for example, Eq. (11).

[5]Several aspects of the problem are considered by A. J. Leggett, Revs. Mod. Phys. 47, 331 (1975). The question of whether such superfluids will be "orbital ferromagnets" was first raised by P. W. Anderson and P. Morel, Phys. Rev. 123, 1911 (1961), and has been recently reconsidered by M. C. Cross, Ref. 3.

[6]According to Wölfle; Ambegaokar, de Gennes, and Rainer; or Cross (Ref. 3) near T_c the coefficient c is just $\hbar\rho_s/8M$. The term in c_0 in the momentum density vanishes in configurations of cylindrical symmetry.

[7]The conclusions that follow also apply to an order parameter whose k-space shape is given by general Y_{lm} ($m \neq 0$) provided that the ^{3}He atomic mass M is replaced by M/m in all formulas. (Note, though, that the quantization condition for a conventional vortex singularity in the irrotational part of $\vec{v}_s$ is not so altered since the order parameter is unchanged by a rotation through $2\pi/m$ about $\vec{l}$.)

[8]This is very close to the original point of view of P. G. de Gennes, Phys. Lett. 44A, 271 (1973).

[9]To derive (6) note that the condition $0 = (\vec{a}_1\cdot\nabla)(\vec{a}_2\cdot\nabla)\vec{\varphi}^\alpha - (1\leftrightarrow 2)$, together with the definition of the $\underline{\Omega}$ tensor ($\vec{a}\cdot\nabla\vec{\varphi}^\alpha = \vec{a}\cdot\underline{\Omega}\times\vec{\varphi}^\alpha$), requires that $0 = [(\vec{a}_1\cdot\nabla)(\vec{a}_2\cdot\underline{\Omega}) - (\vec{a}_2\cdot\nabla)(\vec{a}_1\cdot\underline{\Omega}) - \vec{a}_1\cdot\underline{\Omega}\times\vec{a}_2\cdot\underline{\Omega}]\times\vec{\varphi}^\alpha$, for any two constant vectors $\vec{a}_i$. Since this holds for all three $\vec{\varphi}^\alpha$, the quantity in square brackets must vanish. From this identity (and the use of $\underline{\Omega}$ to eliminate the additional derivatives of $\vec{l}$) it follows at once that $(\vec{a}_1\cdot\nabla)(\vec{a}_2\cdot\underline{\Omega}\cdot\vec{l}) - (1\leftrightarrow 2) = [\vec{a}_2\cdot\underline{\Omega}\times\vec{a}_1\cdot\underline{\Omega}]\cdot\vec{l}$. The terms on the left are proportional to the left-hand side of Eq. (6) [cf. Eq. (4)]. The terms on the right can be rewritten [cf. Eq. (5)] as $[\vec{a}_2\cdot\underline{\hat{\Omega}}\times\vec{a}_1\cdot\underline{\hat{\Omega}}]\cdot\vec{l} = \vec{a}_2\cdot\underline{\hat{\Omega}}\cdot(\vec{a}_1\cdot\nabla)\vec{l}$. But since $(\vec{a}_2\cdot\nabla)\vec{l} = \vec{a}_2\cdot\underline{\Omega}\times\vec{l}$, it follows that $\vec{a}_2\cdot\underline{\hat{\Omega}} = \vec{l}\times(\vec{a}_2\cdot\nabla)\vec{l}$. Hence $[\vec{a}_2\cdot\underline{\Omega}\times\vec{a}_1\cdot\underline{\Omega}]\cdot\vec{l}]\cdot\vec{l} = [(\vec{a}_2\cdot\nabla)\vec{l}\times(\vec{a}_1\cdot\nabla)\vec{l}]\cdot\vec{l}$, which is proportional to the right-hand side of Eq. (6).

[10]See, for example, Cross, Ref. 3. We find that all the the surface terms one must suppress to arrive at (7) from a general set of invariants quadratic in $\vec{v}_s$ and the gradients of $\vec{l}$, can be reduced [with the aid of Eq. (6)] to the single term arising from $\nabla\cdot(\vec{l}\times\vec{v}_s)$, which vanishes for $\vec{l}$ normal to the surface.

[11]Ambegaokar, de Gennes, and Rainer, Ref. 3. The hydrodynamic approximation we make ignores the possible suppression of the amplitude of the order parameter at the surface, due either to diffuse surface scattering (Ambegaokar, de Gennes, and Rainer, Ref. 3) or surface curvature [G. Barton and M. A. Moore, J. Low Temp. Phys. 21, 489 (1975)]. We believe that such effects will alter the angular momentum we calculate by terms that are smaller by a factor of order $\xi(T)/R$, and that therefore they will be of little importance in macroscopic vessels except quite near the transition temperature.

[12]The analogous texture in a nematic liquid crystal has been examined by R. B. Meyer, Philos. Mag. 27, 405 (1973). For a large enough cylinder this texture will have lower free energy than the disgyrations of de Gennes (Ref. 1). W. F. Brinkman informs us that he and P. W. Anderson have described such a texture in ^{3}He-A (to be published), essentially in the representation $\psi \propto \vec{p}\cdot[\hat{r}l_z - \hat{z}l_r + i\hat{\varphi}]e^{i\varphi}$, which avoids explicit reference to $\vec{v}_s$. They also note the order of magnitude of the accompanying angular momentum, but do not give it the interpretation we suggest later.

[13]The factor $(1-l_z)$ removes the singularity at the origin. If the cylinder is externally pressed into very slow rotation it will eventually become favorable for the 1 to change to 2 or 0, through the appearance of a superimposed conventional vortex line. One might also note that the 2 multiplying the mass in this and subsequent formulas is actually the number of atoms in the Cooper n-uple when $m=1$ [and more generally, is n/m, where m is the axial quantum number (see Ref. 7)]. However the 2 extracted from nonlinear ringing [R. A. Webb, R. E. Sager, and J. C. Wheatley, Phys. Rev. Lett. 35, 1010 (1975)] is only a consequence of the Cooper n-uple having a total spin of unity, and is not, contrary to many assertions (for example, Ref. 5, p. 393), direct evidence for $n=2$.

[14]In terms of the order parameter ψ, Eqs. (8) and (9) are the assertion that $\psi \propto \vec{p}\cdot[\hat{r}l_z - \hat{z}l_r + i\hat{\phi}]e^{i\varphi}$ (cf. Ref. 12).

[15]According to Cross (Ref. 3), as T approaches zero, $c \rightarrow \hbar\rho_s/4M$. Thus in this limit the value of $\vec{L}$ is independent of the form of l_z, the decrease in the "intrinsic pair contribution" due to the bending of $\vec{l}$ from the z axis being precisely compensated by the "pair center-of-mass contribution" produced by the circulation (6) required when $\vec{l}$ is nonuniform. It is an interesting question whether this cancelation is an artifact of the cylindrical geometry, or a more general manifestation of the low-temperature behavior of superfluids with $m \neq 0$.

[16]P. G. de Gennes and D. Rainer, Phys. Lett. A46, 429 (1974).