COMMENTS

Integral Formalism of Gauge Fields and General Relativity*

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Yang's new integral theory for gauge fields associated with the group GL(n) is discussed for the particular case when GL(n) is reduced to SL(2, C). It is pointed out that, in this particular case, the theory gives the vacuum Einstein equations, but not the full equations. A modification giving nonvacuum equations consistent with the Einstein theory is pointed out. The field variables are then written as operators.

Because of the large number of theories of gravitation, there should be little interest in finding one more new theory. This is, however, not quite the case when gravitation is obtained from a gauge theory since an important trend in particle physics is in that direction. This is the case of Yang's[1] recent integral formalism of gauge fields associated with the group GL(n) in which he develops gravitational field equations that are related, but not identical, to Einstein's equations. Since Yang's theory is Riemennian covariant, and Riemannian geometry is well understood by now, it is very important to relate the new theory to general relativity theory so as to find out its significance. In this paper I find the conditions under which Yang's theory yields the usual Einstein equations when GL(n) is chosen to be SL(2, C). I then represent both the potential and field as operators in a Hilbert space.

Our starting point is to find out the geometrical meaning of the potential $b_\mu{}^k(x)$ and the field $f_{\mu\nu}{}^k(x)$, both of which are Riemannian covariant. They are related by

$$f_{\mu\nu}{}^k=\frac{\partial b_\mu{}^k}{\partial x^\nu}-\frac{\partial b_\nu{}^k}{\partial x^\mu}-b_\mu{}^i b_\nu{}^j C_{ij}{}^k, \qquad (1)$$

where $f_{\mu\nu}{}^k$ is essentially the Riemann tensor, the superscript k being the group index, and $C_{ij}{}^k$ is the structure constant of the gauge group. The crucial point here is that the potential $b_\mu{}^k(x)$ is not the spinorial affine connection as is the case in Utiyama's[2] gauge theory of gravitation which is an extension of the usual Yang-Mills[3] theory. As is well known, affine connections are not Riemannian covariant, a drawback of the theory. Another crucial difference is that the theory here is Riemannian at the outset, whereas in the Utiyama theory the curvature aspect is introduced in an *ad hoc* way.

Three groups play the most important roles in gravitation: (1) the Lorentz group; (2) the group SL(2, C); and (3) the Poincaré group. Since SL(2, C) is the covering group of the Lorentz group, of interest are left cases (2) and (3). In this paper I choose the gauge group GL(n) to be SL(2, C), i.e., choosing $n=2$ along with demanding that the determinants be equal to 1. The gravitational field is well understood by now in terms of its symmetry and invariance under the group SL(2, C).[4-11] This fact enables us to establish the relation between Yang's new integral formalism for gauge fields and general relativity. The result can be summarized as follows:

When SL(2, C) is taken as the underlying gauge group, the potential $b_\mu{}^k(x)$ becomes complex and $k=1, 2, 3$. Hence one has twelve complex functions. Let l^μ, m^μ, $m^{*\mu}$, and n^μ be four null vectors, satisfying $l_\mu n^\mu = -m_\mu m^{*\mu}=1$, all other products being zero, where l^μ and n^μ are real whereas m^μ is complex. These four vectors can be given a unified notation by putting $l^\mu=\sigma^\mu{}_{00'}$, $m^\mu=\sigma^\mu{}_{01'}$, and $n^\mu=\sigma^\mu{}_{11'}$. The indices ab' of $\sigma^\mu{}_{ab'}$ are dyad indices. The geometrical metric is related to the null tetrad of vectors by $g^{\mu\nu}=\sigma^\mu{}_{ab'}\sigma^{\nu ab'}$, where dyad indices are raised by the Levi-Civita skew-symmetric tensors. With the help of σ^μ one can write the potential in a different representation as $b_{ab'}{}^k=\sigma^\mu{}_{ab'}b_\mu{}^k$. The new functions $b_{ab'}{}^k$ are then the spin coefficient functions, written according to the scheme[5]

$$\begin{aligned} b_{00'}{}^k&=(-\kappa, \epsilon, \pi), \quad b_{01'}{}^k=(-\sigma, \beta, \mu),\\ b_{10'}{}^k&=(-\rho, \alpha, \lambda), \quad b_{11'}{}^k=(-\tau, \gamma, \nu). \end{aligned} \qquad (2)$$

A similar representation can be given to the field by introducing the quantities $f_{ab'cd'} = \sigma^{\mu}{}_{ab'}\sigma^{\nu}{}_{cd'}f_{\mu\nu}$. The new functions describe all irreducible components of the Riemann tensor according to the scheme[5]

$$f_{01'00'}{}^{k} = (-\psi_0, \psi_1, \psi_2 + 2\Lambda), \quad f_{11'10'}{}^{k} = (-\psi_2 - 2\Lambda, \psi_3, \psi_4), \quad f_{10'00'}{}^{k} = (-\varphi_{00}, \varphi_{10}, \varphi_{20}),$$

$$f_{11'01'}{}^{k} = (-\varphi_{02}, \varphi_{12}, \varphi_{22}), \quad f_{11'00'}{}^{k} = (-\psi_1 - \varphi_{01}, \psi_2 + \varphi_{11} - \Lambda, \psi_3 + \varphi_{21}), \tag{3}$$

$$f_{10'01'}{}^{k} = (\psi_1 - \varphi_{01}, -\psi_2 + \varphi_{11} + \Lambda, -\psi_3 + \varphi_{21}).$$

Here $\psi_0, \ldots, \psi_4$ describe the Weyl tensor, the φ's describe the trace-free part of the Ricci tensor, and $\Lambda = R/24$, where R is the Ricci scalar.

Now we come to the important problem of the field equation. Yang's Langrangian is of the form $\mathfrak{L} = \sqrt{-g} f_{\mu\nu}{}^{k} f^{\mu\nu}{}_{k}$, where space-time indices were raised by the geometrical metric, and group indices by the group metric.[12] Yang correctly states that this Lagrangian does not yield the Einstein field equations. It has been shown,[7] however, that $\mathfrak{L}$ gives[13] the vacuum Einstein field equations written in the formalism of Newman and Penrose when the group SL(2, C) is the gauge group; the Euler-Lagrange equation leads to

$$\partial^{cd'}F_{ef'cd'} - \{\delta_{f'}{}^{b'}(B^{cd'})^{a}{}_{e} + \delta^{a}{}_{e}(B^{\dagger d'c})_{f'}{}^{b'}\}F_{ab'cd'} + \{B_{p}{}^{d'})^{cp} + (B_{q'}{}^{\dagger c})^{q'd'}\}F_{ef'cd'} - [B^{cd'}, F_{ef'cd'}] = 0, \tag{4}$$

and Eq. (1) gives $(\partial_{ab'} = \sigma^{\mu}{}_{ab'}\partial_{\mu})$

$$\partial_{cd'}B_{ab'} - \partial_{ab'}B_{cd'} - (B_{cd'})_{a}{}^{f}B_{fb'} - (B_{d'c}{}^{\dagger})^{f'}{}_{b'}B_{af'} + (B_{ab'})_{c}{}^{f}B_{fd'} + (B_{b'a}{}^{\dagger})^{f'}{}_{d'}B_{cf'} + [B_{ab'}, B_{cd'}] = F_{ab'cd'}. \tag{5}$$

Equations (4) and (5) are identical to Eqs. (4.5) and (4.2) of Ref. 6. The matrices $F_{ab'cd'} = \sum f_{ab'cd'}{}^{k} \times g_k$ and $B_{ab'} = \sum b_{ab'}{}^{k} g_k$, where g_k are the infinitesimal matrices of the group SL(2, C). The Lagrangian $\mathfrak{L}$ does not give a third set of equations, Eqs. (6.10) of Ref. 6, that relate $\sigma^{\mu}{}_{ab'}$ to $b_{\mu}{}^{k}$. This third set, however, can easily be obtained separately.[7] To obtain a Lagrangian that gives the nonvacuum field equations one notices that $\mathfrak{L}$ has the form $\sqrt{-g}\,\sigma^4 f^2$. The only alternative way of writing such a combination of σ and f is in the form[14] $\tilde{\mathfrak{L}} = \sqrt{-g}\,\sigma^{\mu}{}_{ab'}\sigma^{\nu}{}_{cd'}\sigma^{\alpha ad'}\sigma^{\beta cb'}f_{\mu\nu}{}^{k}f_{\alpha\beta k}$. The Lagrangian $\tilde{\mathfrak{L}}$ was shown[8,15] to lead to two sets of field equations with matter. Using the totally skew-symmetric tensor $\epsilon^{\mu\nu\alpha\beta}$ of weight +1 whose components are 1, 0, −1, one obtains for $\tilde{\mathfrak{L}}$ the expression $\tilde{\mathfrak{L}} = \epsilon^{\mu\nu\alpha\beta}f_{\mu\nu}{}^{k}f_{\alpha\beta k}$. Hence $\tilde{\mathfrak{L}}$ is free of the metric tensor that occurs in $\mathfrak{L}$. In fact, $\tilde{\mathfrak{L}}$ is the only Lagrangian that satisfies this important property and may thus be considered as a functional of the potential B and field F, just like other non-Abelian gauge theories.

We conclude our remarks by giving operator versions to the potential and field.[16] This can be done in a remarkably natural way in the present context of gravitational theory. To this end we define the matrices $B_{\mu} = b_{\mu}{}^{k}g_k$ and $F_{\mu\nu}{}^{k}g_k$, where g_k are the infinitesimal matrices of the group. If $g \to D(g)$ is a representation of the group, then g_k go, under the representation, into the infinitesimal operators D_k, as $g_k \to D_k$, where D_k are defined in the space representations. Now define the operator functions

$$\hat{B}_{\mu} = \sum_k b_{\mu}{}^{k}D_k, \quad \hat{F}_{\mu\nu} = \sum_k f_{\mu\nu}{}^{k}D_k, \tag{6}$$

and denote $b_{\mu}{}^{k}(x)D_k$ by $\hat{b}_{\mu}{}^{k}$ and $f_{\mu\nu}{}^{k}(x)D_k$ by $\hat{f}_{\mu\nu}{}^{k}$. The operator functions $\hat{b}_{\mu}{}^{k}$ and $\hat{f}_{\mu\nu}{}^{k}$ then describe the gravitational field. Since $g \to D(g)$ is a representation, the operators D_k satisfy the same Lie algebra as the infinitesimal matrices g_k. Hence one has for the commutation relations of the field operators the following:

$$[\hat{b}_0{}^{i}(x), \hat{b}_0{}^{j}(y)]_{x_0 = y_0} = i\hbar C^{ij}{}_{k}\hat{b}_0{}^{k}(x)\delta^3(x-y). \tag{7}$$

Equation (7) has the same structure as that used by Lee, Weinberg, and Zumino.[17] Commutation relations of other components can be found and their implications on quantum gravidynamics should be further explored.

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[12]When the gauge group is restricted to the group SL(2, C), Yang's Lagrangian is equal to $\sqrt{-g}\,\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})$, where $F_{\mu\nu}$ is the matrix $F_{\mu\nu}=\sum f_{\mu\nu}{}^{k}g_k$, and g_k are the infinitesimal matrices of the group SL(2, C).

[13]Although the equations of motion obtained involve the full Riemann tensor, obviously the Riemann tensor $f_{\mu\nu}{}^{k}$ should then be taken according to Eq. (3) with $\varphi_{ij}=\Lambda=0$.

[14]The Riemann tensor $f_{\mu\nu}{}^{k}$ is taken now according to Eq. (3) with $\varphi_{ij}\simeq T_{ij}$ and $\Lambda=T/24$, where T_{ij} is the energy-momentum tensor. That does not mean that the Ricci tensor part of the Riemann tensor is taken fixed. It is still a dynamical variable. It should be noted that in this paper no attempt is made to obtain the full set of field equations of the gravitational theory from a variational principle, such as Eqs. (20) to (22) of Ref. 1, when considering gravitation as a gauge theory. No attempt is also made, in fact, to discuss Yang'g gravitational Lagrangian at all [see, for example, A. H. Thompson, Phys. Rev. Lett. 34, 507 (1975)]. For example, among the equations that are *not* obtained in our theory are $R_{\mu\nu}=0$ or $R_{\mu\nu}=T_{\mu\nu}-\frac{1}{2}g_{\mu\nu}T$ (formally the Einstein equations), although these equations are not as central in the Newman-Penrose scheme as they are in the usual presentation of general relativity theory. However, it is worthwhile emphasizing that the Newman-Penrose dynamical equations (Bianchi equations and the definition of the Riemann tensor in terms of the spin coefficients) *are* obtained from our variational principle.

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Long-Range Interactions in Semiconductors*

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We derive an exponentially decaying interaction between atoms in semiconductors. The decay length is $\hbar[2(m_1+m_2)E_g]^{-1/2}$, where m_1 and m_2 are the valence-band and conduction-band effective masses and E_g is the minimum energy gap. We show that Weber's bond-charge model of lattice vibrations leads to exponentially decaying interactions. Comparing the experimental vibration spectrum and the theoretical decay length suggests that the flattening of the TA mode is due to this long-range interaction.

Weber[1] has extended Phillips's[2] bond-charge (BC) model of lattice vibrations in semiconductors to explain the characteristic flattening of the transverse acoustic phonon mode away from the zone center and the unusually low frequency of this mode at the zone boundary. To illustrate the model Weber considers a monatomic linear chain with lattice constant d where each atom is coupled to its nearest-neighbor BC by a force constant f, and the nearest-neighbor BC's are coupled with a force constant f'. A simple calculation gives the dispersion relation

$$M\omega^2=2f\frac{(f+2f')\sin^2(\pi k/2k_0)}{f+2f'\sin^2(\pi k/2k_0)}, \qquad (1)$$

where M is the atomic mass; k is the wave number of the mode and k_0 its value π/d at the zone boundary. (The mass of the BC's is set equal to zero.)

This model can be described equally well in terms of a Born–von Kármán expansion in which only two-body interactions between atoms are considered. The BC in Weber's model produces long-range forces. This can be seen by displacing atom i while keeping the positions of the other atoms fixed; one can then calculate the force transmitted through the BC's to atom j by minimizing the total energy of the distorted lattice with respect to the BC positions. This calcula-