PHYSICAL REVIEW LETTERS

VOLUME 36 1 MARCH 1976 NUMBER 9

Unified, Relativistic Quark Model for Mesons

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(Received 24 June 1975)

A relativistic wave equation is solved for equal-mass quarks interacting via a combined linear and "Coulombic" potential, the latter being softened at short distances in accord with asymptotic freedom. Assuming that colored SU(4) is broken only by the quark mass differences, I predict masses and leptonic decay widths for the ρ, φ, and ψ meson families of states. Agreement with experiment is excellent.

In 1972, I argued that the tension characterizing the hadronic medium is *independent* of the amount by which it has been stretched.[1] In this view, the work done in separating two quarks is used to create enough particle-antiparticle pairs to maintain a uniform chain linking the quarks. This leads to an effective potential which, in the static limit, is proportional to the quark separation (linear potential).

The physical picture of uniform chains linking quarks has been refined and advocated by Kogut and Susskind (and subsequently by many others) as a mechanism for quark confinement.[2] The quarks are regarded as sources and sinks of colored Yang-Mills gluons, and the gluon flux is confined to tubes of uniform cross section by nonlinearities in the field equations. In addition to the linear potential, theory suggests[2] an attractive $1/r$ potential, with effective coupling which tends to zero at short distances, i.e., large momentum transfer (asymptotic freedom).[2,3] The gluon interactions are assumed to be independent of quark "flavor," so that strong-interaction symmetry is broken only by quark mass differences.[4]

While the preceding ideas are highly attractive, confrontation with experiment has been fragmentary and indecisive. It is therefore desirable to construct a unified model which is realistic enough to be indicative, yet simple enough to be solvable.

In this spirit, we consider a relativistic wave equation for two equal-mass quarks interacting via the potential

$$V(r) = Tr - \alpha_s(r)/r, \quad (1)$$

$$\alpha_s(r) = \alpha_s(\infty)[1 - \exp(-r/r_0)], \quad (2)$$

where T denotes the tension along the flux tube, α_s denotes the "fine-structure constant" of strong interactions, and the term in square brackets in Eq. (2) represents a simple and tractable way of softening the $1/r$ potential for distances $r < r_0$, where r_0 remains to be determined.[5]

We shall work in the center-of-mass frame, in units with $\hbar = c = 1$. For simplicity, we neglect quark spin and the possibility of virtual-quark creation (e.g., vacuum polarization). A state is then described by a one-component wave function $\Psi(\vec{r}, t)$, where $\vec{r}$ denotes the relative coordinate. The Hamiltonian is

$$\mathcal{H} = 2(p^2 + m^2)^{1/2} + V(r), \quad (3)$$

with $p \equiv |\vec{p}|$.

Interpretation of the kinetic term in $\mathcal{H}$ is facilitated by use of momentum space, where we consider the eigenfunctions

$$\tilde{\Psi}_{nlm}(\vec{p}, t) = \tilde{u}_{nl}(p) Y_{lm}(\theta_p, \varphi_p) \exp(-iE_{nl}t).$$

The eigenvalue equation may then be written as

$$[E_{nl} - 2(p^2+m^2)^{1/2}]\,\tilde{u}_{nl}(p) = \int_0^\infty dp' p'^2 K_l(p,p')\tilde{u}_{nl}(p'), \tag{4}$$

where

$$K_l(p,p') \equiv (2/\pi)\int_0^\infty dr\, r^2 V(r) j_l(pr) j_l(p'r),$$

with j_l denoting the spherical Bessel function of order l. The kernels K_l have been evaluated[6] for $0 \le l \le 4$. Explicit results will be presented elsewhere, together with the numerical techniques which have been used to obtain the E_{nl} and $\tilde{u}_{nl}(p)$.

We note here that in the limit of large, circular orbits, the relation between net energy E and angular momentum $[l(l+1)]^{1/2}$ is readily obtained by classical arguments. We find in this limit that the quark velocities approach unity, and that[7]

$$l \to (8T)^{-1}E^2 + \tfrac{1}{2}[\alpha_s(\infty) - 1 - (2T)^{-1}m^2]. \tag{5}$$

Hence leading Regge trajectories are linear functions of E^2 in the asymptotic region, with slope $\alpha' = (8T)^{-1}$. For *low* energies, however, the trajectories need *not* be linear, since the classical limit is not applicable for small orbits.

A primary goal of the present work is to test the hypothesis that strong-interaction symmetry is broken only by quark mass differences. We consider four quarks here, whose masses will be denoted by m_u, m_d, m_s, and m_c, with $m_u = m_d$. The fourth quark may be regarded as the charmed quark of SU(4), but this identification is not essential. Since the present formalism is restricted to the case of equal-mass constituents,[8] the candidates for study are the ρ meson family ($u\bar{u} - d\bar{d}$), the φ meson family ($\sim s\bar{s}$), and tentatively, the ψ meson family ($c\bar{c}$). We shall regard φ mesons as pure $s\bar{s}$ when computing their masses, but use a singlet-octet mixing angle of 40°[9] when computing their leptonic decay widths. Pseudoscalar mesons will not be considered here, because of the complications arising from their potential roles as Goldstone bosons (they may be collective excitations rather than di-quark systems).

The present model will be tested in two ways.[10] One obvious requirement is that the observed mass spectrum should be reproduced. In addition, we shall compute leptonic decay widths for the S states (we assume quark spins to be parallel in the ρ, φ, and ψ families, so that S states are $^3S_1$, and decay into e^+e^- and $\mu^+\mu^-$ is permitted). The leptonic decay widths are given by[11]

$$\Gamma_{\text{lep}} \simeq (16\pi/3)e^4(C^2/M^2)F^2, \tag{6}$$

where e denotes the electron charge ($e^2 \simeq 1/137$), C depends on the quark content of the decaying meson [$C^2 = \frac{3}{2}$, $\frac{1}{2}\cos^2\theta_{\text{mix}}$, and $\frac{4}{3}$ for the ρ, φ, and ψ mesons, respectively, assuming colored SU(4), with θ_{mix} denoting the singlet-octet mixing angle], M denotes the meson mass, and

$$F = (2\pi)^{-3/2}\int d^3p\, \tilde{\Psi}(\vec{p}) f(\vec{p}). \tag{7}$$

For a nonrelativistic bound state, the function $f(\vec{p})$ appearing in Eq. (7) is unity, and $F = \Psi(0)$. For relativistic bound states, however, $f(\vec{p})$ is not presently known. Some insight may be gained by noting that for a Coulomb potential (e.g., in a $\mu^+\mu^-$ bound state, which also decays into e^+e^-), $\tilde{\Psi}(\vec{p})$ falls off so slowly for large p that with $f=1$, the integral in Eq. (7) diverges [i.e., $\Psi(0) = \infty$]. In order for the integral to converge, it may be shown that $f(\vec{p})$ must fall off at least as rapidly as $p^{-(1/2+\epsilon)}$ for some $\epsilon > 0$.[12] Hence a plausible guess (with the correct nonrelativistic limit) is that

$$f(\vec{p}) \simeq m/(p^2+m^2)^{1/2}, \tag{8}$$

where m denotes the quark mass. We shall use Eq. (8) in the predictions for Γ_{lep}.[10]

We are now in a position to examine the results of numerical computations. Because of the large number and diversity of experimental results we seek to reproduce, it is difficult to ascertain the truly optimal values for the parameters of the present model. I find, however, that the model is in very good agreement with experiment if we assume that $T = 0.1585$ GeV^2, $\alpha_s(\infty) = 2.685$, $r_0 = 2.0$ GeV^{-1}, $m_u = m_d = 0.292$ GeV, $m_s = 0.479$ GeV, and $m_c = 1.692$ GeV.

All the experimentally known masses and leptonic decay widths in the ρ, φ, and ψ families are shown in Table I, together with the present predictions for these quantities. Note that masses are in extremely good agreement, and that the first radial excitation of the $\rho(770)$ occurs at 1515 MeV, in agreement with the observed $\rho'(\sim 1550)$. Hence the present model indicates that the experimentally elusive $\rho'(1270)$ does not in fact exist.

With regard to leptonic decay widths, the agreement in Table I ranges from excellent to fairly good. Furthermore, the states for which the agreement is least good are precisely those

TABLE I. Predicted and observed values for masses and leptonic decay widths of states in the ρ, φ, and ψ meson families. The $\chi(3410)$, $\chi(3530)$, and $P_c(3500)$ are regarded as a P-wave spin triplet, with average mass of 3480 MeV. Data are taken from Refs. 13 and 14.

	Mass (MeV)		Γ_{lep} (keV)	
	Theory	Observed	Theory	Observed
ρ	777	770 ± 10	7.0	7.0 ± 1.2
ρ'	1515	~ 1550	0.7	?
A_2	1298	1310 ± 10	...	...
g	1701	1686 ± 20	...	...
φ	1021	1019.7 ± 0.3	1.7	1.3 ± 0.1
f'	1513	1516 ± 3	...	...
ψ	3086	3095 ± 4	4.9	4.8 ± 0.6
ψ'	3694	3684 ± 5	2.4	2.1 ± 0.3
ψ''	4131	~ 4100	1.6	1.8 to 3.3
ψ'''	4485	~ 4450	1.1	0.4 to 0.8
χ, P_c	3477	~ 3480	...	...

states with conspicuous ambiguities. For the $\varphi(1019)$, singlet-octet mixing undermines the computation of $\tilde{\Psi}(\vec{p})$. The $\psi''(4100)$ is presumably near the threshold for production of charmed meson pairs, and threshold effects may contribute to the experimental ψ'' peak in e^+e^- scattering. Another complication for the ψ family is that S-D mixing seems likely to occur above 4 GeV (as will be discussed below).[15]

I note here that the present model has passed a theoretical test, in that the α_s required to fit the data is *positive* (i.e., the "Coulomb" force is *attractive*), in accord with theoretical expectations.[2]

The striking success of the model in simultaneously reproducing eleven masses and six leptonic decay widths in the ρ, φ, and ψ families gives strong support to the underlying assumptions. In particular, I conclude that all members of the ψ family in Table I have identical quark content, and that the effective[16] charge of "charmed" quark is $\pm\frac{2}{3}e$ (as assumed here in predicting leptonic decay widths).

Table II presents a more complete set of predictions for the radial and orbital excitations in the ρ, φ, and ψ families. Note that in the ψ family, the lowest five D states are separated from the nearest S states by gaps of 94, 60, 43, 33, and 26 MeV, respectively. The largest of these gaps is for the state below 4 GeV, where ψ widths are narrow and energies are well defined. Hence S-D mixing is least likely to occur for this state, in accord with the experimental absence of any

TABLE II. Predicted masses in MeV, and leptonic decay widths for $l=0$ states in keV, for the ρ, φ, and ψ meson families.

Meson	Masses $l=0$	$l=1$	$l=2$	$l=3$	$l=4$	Γ_{lep}
ρ	777	1298	1701	2036	2327	7.0
	1515	1854	2157	2430	2678	0.7
	2041	2299	2548	2780	2994	0.3
φ	1021	1513	1897	2218	2498	1.7
	1739	2060	2348	2609	2847	0.3
	2247	2495	2732	2954	3162	0.1
ψ	3086	3477	3788	4053	4285	4.9
	3694	3956	4191	4406	4602	2.4
	4131	4335	4528	4712	4884	1.6
	4485	4658	4825	4987	...	1.1
	4792	4943	5092	...	...	0.8
	5066	...	...	...	...	0.6

additional peaks in e^+e^- scattering near the $\psi'(3684)$.[17] Above 4 GeV, the gaps between S and D states become increasingly smaller, and *all are smaller than the intrinsic resolution of resonance energies in this region.* Hence S-D mixing is likely to be appreciable for states above 4 GeV, in accord with the experimentally apparent structure[13] within the $\psi''(4100)$ and $\psi'''(4450)$ peaks in e^+e^- scattering.

While all leading trajectories in the present model have asymptotic slopes of $\alpha' = (8T)^{-1} = 0.79$ GeV^{-2}, the slope corresponding to the first two states on the ψ trajectory is only 0.39 GeV^{-2}. Hence this trajectory has appreciable curvature, which could resolve a puzzle noted by Finkelstein regarding the ψp total cross section.[18]

The final column in Table II displays a more complete set of predictions for leptonic decay widths. Additional experimental work on the widths of radially excited states is clearly desired. It is also important to obtain a firmer theoretical basis for the $f(\vec{p})$ of Eq. (8). To indicate the role played by Eq. (8) in the present work, I remark that if we had instead used the nonrelativistic value $f=1$, the leptonic widths of the $\rho(770)$, $\varphi(1019)$, and $\psi(3095)$ would have been larger by factors of 6.7, 3.7, and 1.5, respectively, than shown in Tables I and II. Hence Eq. (8) is crucial to the success of the predictions for leptonic widths.

In summary, the present results lend strong support to a simple quark model for the ρ, φ, and ψ meson families of states, with a group structure consistent at this level of phenomenology with colored SU(4). The quarks should be regarded as bona fide physical objects, since the present dynamical framework is based on this assumption. An effective potential similar to that described by Eqs. (1) and (2) also receives strong support from the present work. It seems appropriate to conclude that a fundamental understanding of meson phenomenology lies close at hand.

Informative discussions with J. Kogut, E. Eichten, and H. Pagels are gratefully acknowledged.

[1]E. P. Tryon, Phys. Rev. Lett. 28, 1605 (1972). At the time of this work, the experimental absence of quarks was regarded as evidence that free quarks were more massive than other hadrons then observed, and much more massive than (e.g.) a ρ meson. In that case, the binding energy in a ρ would nearly equal (in order to cancel) the masses of the constituent q and $\bar{q}$. With such a near cancelation between potential energy and mass energy, the author visualized that pulling apart the original q and $\bar{q}$ in a ρ would result in the generation of a uniform chain of $q\bar{q}$ pairs linking the original q and $\bar{q}$, giving rise to an effective linear potential (uniform energy per unit length of the chain). For reasons too complex and unfortunate to describe here, the author conjectured that the orbital excitations would be spinning toroids rather than open strings. The great contribution of Susskind and a series of co-workers was to recognize that if the chain consisted of some conserved flux instead of $q\bar{q}$ pairs, then even light quarks could be absolutely confined, and excited states would be open strings (as had been suggested by earlier studies of duality). A series of models with conserved flux was investigated, beginning with $(1+1)$-dimensional quantum electrodynamics [A. Casher, J. Kogut, and L. Susskind, Phys. Rev. Lett. 31, 792 (1973)], and culminating in the theory of colored Yang-Mills gluons described in J. Kogut and L. Susskind, Phys. Rev. D 9, 3501 (1974). Kogut referred to the linear potential between quarks as "Tryon's law" [J. Kogut, in *Particles and Fields—1974*, edited by C. E. Carlson, AIP Conference Proceedings No. 23 (American Institute of Physics, New York, 1975)], but an equally appropriate name would be the "Tryon-Susskind law."

[2]Kogut and Susskind, Ref. 1.

[3]Cf. H. D. Politzer, Phys. Rev. Lett. 30, 1346 (1973); D. J. Gross and F. Wilczek, Phys. Rev. Lett. 30, 1343 (1973).

[4]An excellent review is given by S. Glashow, Sci. Am. 233, No. 4, 38 (1975).

[5]Equation (2) probably exaggerates the softness of the potential at short distances (large momentum transfer). This should only be significant for S waves, however, and is partly compensated by the present use of a spin-0 wave equation. In particular, S-wave solutions to the Klein-Gordon equation are known to be more tightly bound in a Coulomb potential than are solutions to be Dirac equation, because the spin *Zitterbewegung* reduces the probability for a Dirac particle to be localized at the origin.

[6]The kernels diverge because of the linear term in V. This difficulty is merely technical, however, and has been overcome by a sequence of limiting procedures (E. P. Tryon, to be published).

[7]The *intercept* of this asymptotic expression is uncertain by a term of order unity (i.e., $\hbar$), because the classical limit may differ from the quantum analog by such a term.

[8]For unequal-mass constituents, there exists no *a priori* representation for the center-of-mass coordinate, since the effective mass of each constituent depends on its (*a priori* unknown) momentum.

[9]N. P. Samios *et al.*, Rev. Mod. Phys. 46, 49 (1974).

[10]Some earlier authors predicted rates for γ-ray transitions in charmonium, obtaining results much larger than are observed. It has recently been noted, however (E. P. Tryon, to be published), that such predictions involved implicit assumptions which are not tenable for charmonium, and that a proper calculation would require a more detailed knowledge of dynamics than is presently available. Hence I do not attempt such a calculation here.

[11]Cf. R. van Royen and V. F. Weisskopf, Nuovo Cimento 50, 617 (1967). Equation (6) is valid for either electrons or muons, within a factor $1+O(m_{\text{lep}}{}^4/M^4)$.

[12]This remark applies to solutions of Eq. (4) with a pure Coulomb potential. Analogous solutions to the Dirac equation are also known to be infinite at the origin of coordinate space, but I have not established the precise rate at which $f(\vec{p})$ would have to vanish for large p in order to make F finite for the Dirac equation.

[13]Cf. H. Harari, in Proceedings of the International Symposium on Lepton and Photon Interactions at High Energies, Stanford, California, August 1975 (to be published), and references cited therein; F. J. Gilman, *ibid.*

[14]A. M. Boyarski *et al.*, Phys. Rev. Lett. 34, 1357 (1975); G. J. Feldman *et al.*, Phys. Rev. Lett. 35, 821 (1975); V. Lüth *et al.*, Phys. Rev. Lett. 35, 1124 (1975); W. Tanenbaum *et al.*, Phys. Rev. Lett. 35, 1323 (1975); V. Chaloupka, Phys. Lett. 50B, 1 (1974).

[15]Note, however, that S-D mixing should lead to the same *integrated* (over energy) cross section for e^+e^- scattering as would be the case without such mixing.

[16]If there is more than one flavor of quark in a ψ, the effective charge means (e.g.) the sense in which the quark superposition $(u\bar{u}-d\bar{d})$ has an effective charge of $\frac{1}{2}e$.

[17]A. M. Boyarski *et al.*, Phys. Rev. Lett. 34, 762 (1975).

[18]J. Finkelstein, Phys. Rev. D 11, 3337 (1975).