ticles may affect the probability of chunk emission during proton bombardment. Moreover, in view of the thin targets and relatively smooth surface finish, the present values are not inconsistent with later results of Kaminsky,[10] who reported that the yield of chunks decreases greatly as the surface finish is improved. The sputtering yields for high-energy protons measured in the present experiment are more than an order of magnitude lower than the atomic deposit measured by Kaminsky, Peavey, and Das,[1,2] and are consistent with the range of values obtained in recent fast-neutron measurements by Harling *et al.*[3]

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*Work supported in part by the National Science Foundation Grants No. MPS71-02670 A05 and No. GP-43585 and the Ford Motor Company.

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COMMENTS

Photon-Photon Scattering Contribution to the Anomalous Magnetic Moment of the Muon*

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A careful and systematic recomputation of the photon-photon scattering contribution to the muon magnetic-moment anomaly is made. The result is $\Delta a_{\text{ph-ph}} = (21.32 \pm 0.05)(\alpha/\pi)^3$ leading to the theoretical value $a_\mu^{\text{th}} = (1165\,918 \pm 10) \times 10^{-9}$.

It has been known[1] for several years that the photon-photon scattering contribution dominates the sixth-order muon magnetic-moment anomaly. The computed results[1-4] obtained, however, disagree with each other well outside of their assigned 91% confidence levels:

$$\frac{\Delta a_{\text{ph-ph}}}{(\alpha/\pi)^3} = \begin{cases} 18.4 \pm 1.1, & \text{Ref. 1,} \\ 20.77 \pm 0.43, & \text{Ref. 2,} \\ 19.76 \pm 0.16, & \text{Ref. 3,} \\ 19.79 \pm 0.16, & \text{Ref. 4.} \end{cases} \tag{1}$$

In view of this situation and the continuing rapid increase in the precision of the experimental value,[5] it is highly desirable to resolve this problem and obtain an accurate value of this contribution which can be viewed with confidence.

In Feynman parametric form one can express this contribution as an integral over a seven-dimensional simplex:

$$\frac{\Delta a_{\text{ph-ph}}}{(\alpha/\pi)^3} \equiv I_0 = \int_0^1 dz_1 \ldots \int_0^1 dz_8\, F(z_1, \ldots, z_8)\delta(1-z_T), \quad (2)$$

where

$$z_T = \sum_{i=1}^{8} z_i .$$

We decided to use the function $F(z)$ of Aldins *et al.*[1,6] and investigate, in a careful and systematic way, the numerical integration procedure involved.

The previous difficulty will be shown to be due to the singularity which the integrand possesses in the four-dimensional region V_4: $z_6 = z_7 = z_8 = 0$. The integrand is not square integrable. This has the effect of causing the integral to be systematically underestimated, and, at the same time, providing an error estimate which is overly optimistic.[7]

We make a transformation into the seven-dimensional hypercube:

$$\frac{\Delta a_{\text{ph-ph}}}{(\alpha/\pi)^3} = \int_0^1 d\alpha_1 \ldots \int_0^1 d\alpha_7\, f(\alpha_1, \ldots, \alpha_7). \quad (3)$$

To investigate the significance of the region V_4 for Eq. (2) we define

$$I(\epsilon) = \int_0^1 dz_1 \ldots \int_0^1 dz_5 \int_\epsilon^1 dz_6 \int_\epsilon^1 dz_7 \int_\epsilon^1 dz_8\, F(z_1, \ldots, z_8)\delta(1-z_T), \quad (4)$$

and the quantity of interest will be given by $I_0 = I(0)$. Figure 1 clearly shows the importance of the region V_4. The computations were done in the hypercube as indicated in Eq. (3) using SPCINT.[8] The calculations reported in this Letter required approximately 15 h of CPU time on the Stanford Linear Accelerator Center tripak coupled system.

As expected, $I(\epsilon)$ is readily evaluated accurately if ϵ is not too small, but the statistical error increases substantially as $\epsilon \to 0$. By carefully studying the dominant behavior of $I(\epsilon)$ it is seen that

$$I(\epsilon) \sim I_0 - A\sqrt{\epsilon}, \quad (5)$$

as $\epsilon \to 0$, with $A \sim 100$. Hence, as one method of evaluating I_0, we compute $I(\epsilon)$ accurately for values of ϵ small enough to see this asymptotic behavior and then extrapolate to $\epsilon = 0$. The results are shown in Fig. 2. Besides the visual extrapolation, Padé approximants[9,10] (type II) were used to do the extrapolation, yielding[11]

$$I_0 = 21.33 \pm 0.07. \quad (6)$$

As an independent check, we have evaluated (also in the hypercube)

$$I(\epsilon_1, \epsilon_2) \equiv \int_{V'} dz\, F(z)\delta(1-z_T) = I(\epsilon_1) - I(\epsilon_2), \quad (7)$$

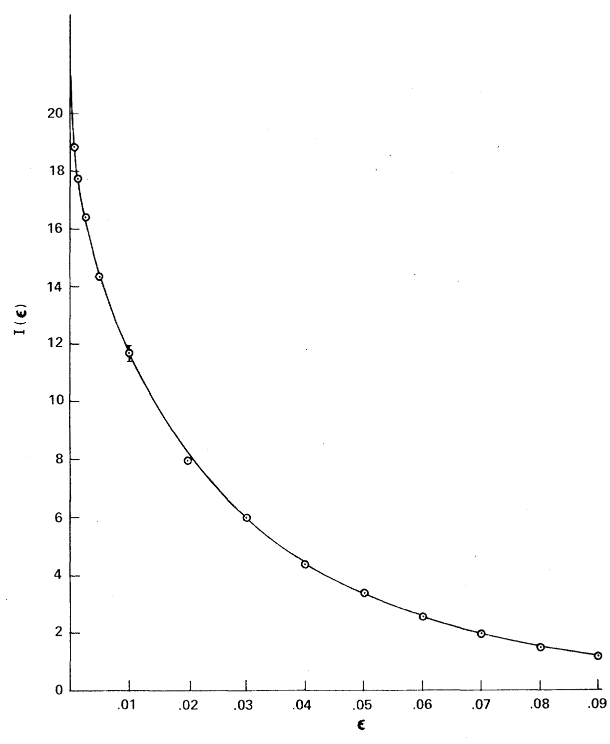

FIG. 1. The function $I(\epsilon)$ versus ϵ. The curve goes to zero at $\epsilon = \frac{1}{3}$ as $(\frac{1}{3} - \epsilon)^6$. The error bars in Figs. 1–4 represent 80% confidence levels. (No error bars are shown when they are inside the circle about the point.)

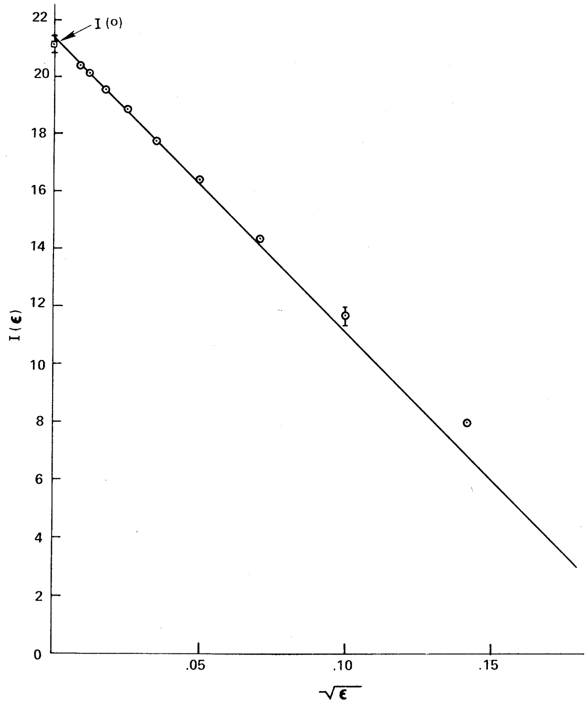

FIG. 2. $I(\epsilon)$ versus $\sqrt{\epsilon}$ for small ϵ. The linear visual extrapolation is shown. Also shown is the value given in Eq. (12), obtained from the direct evaluation of $I(0)$.

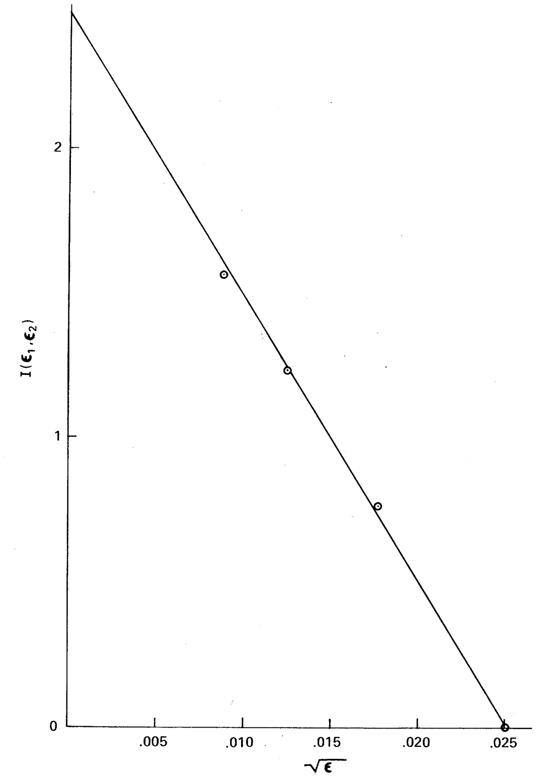

FIG. 3. The function $I(\epsilon_1,\epsilon_2)$ versus $\sqrt{\epsilon_1}$ for small ϵ_1 and $\epsilon_2 = 0.625\times 10^{-3}$. A visual linear fit which goes through zero at $\epsilon_1=\epsilon_2$ is shown.

with $\epsilon_2 = 0.625\times 10^{-3}$, where V' is the region given by ($z_6 > \epsilon_1$ *and* $z_7 < \epsilon_1$ *and* $z_8 > \epsilon_1$) *and* ($z_6 < \epsilon_2$ *or* $z_7 < \epsilon_2$ *or* $z_8 < \epsilon_2$). Figure 3 shows that the results for $I(\epsilon_1, \epsilon_2)$ are consistent with a straight line *which goes through zero at* $\epsilon_1 = \epsilon_2$ *and has the same slope* as the line of Fig. 2, yielding

$$I_0 = 21.3 \pm 0.2. \tag{8}$$

We then decided to evaluate $I(0,\epsilon_2)$ very accurately, obtaining

$$I(0,\epsilon_2) = 2.48 \pm 0.05. \tag{9}$$

Combining this with

$$I(\epsilon_2) = 18.82 \pm 0.06, \tag{10}$$

we obtain the extremely accurate result

$$I_0 = I(0,\epsilon_2) + I(\epsilon_2) = 21.30 \pm 0.08. \tag{11}$$

The agreement with Eqs. (6) and (8) is very gratifying.

As a further check we have evaluated I_0 directly from Eq. (3). The results obtainable are much less accurate but the value

$$I(0) = 21.1 \pm 0.3, \tag{12}$$

is consistent with the previously obtained results.

The increased familiarity with the properties of the integrand acquired from these computations led to one more change of variables, which we used as a final check. We define

$$D(\epsilon) \equiv 32\int_\epsilon^1 dT \int_1^2 dR' \int_1^2 dS' \int_0^1 dX \int_0^1 dY \int_0^1 dU \int_0^1 dV \times [YVT^2RS(2-S')(2-R')/R'^3S'^3]\,\theta(1-Y-V-T)F(z_1,\dots,z_8),$$

with

$$\begin{aligned} &z_1 = Y(1-X),\quad z_2 = XY,\quad z_3 = 1-Y-V-T,\quad z_4 = UV,\quad z_5 = V(1-U),\\ &z_6 = RS^2T,\quad z_7 = R(1-S^2)T,\quad z_8 = (1-R)T,\quad R = 4(R'-1)/R'^2,\quad S = 4(S'-1)/S'^2. \end{aligned} \tag{13}$$

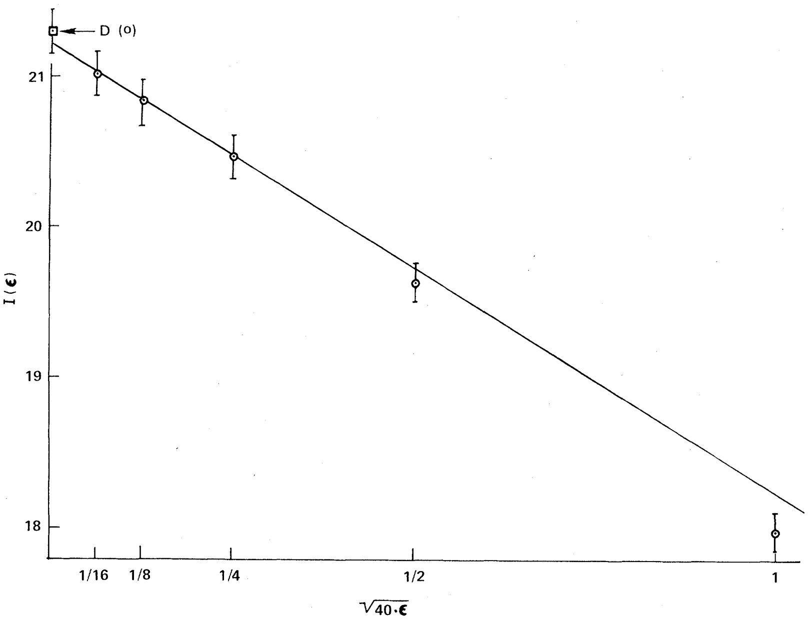

FIG. 4. $D(\epsilon)$ versus $\sqrt{\epsilon}$ (in units of $1/\sqrt{40}=0.158\,114$). The linear visual extrapolation is shown, as well as the value given in Eq. (16), obtained from the direct evaluation of $D(0)$.

We then transform (Y, V, T) into the three-dimensional unit cube. Again, we are interested in the extrapolation $D(0)=I_0$. The behavior for small ϵ is relatively easy to determine. We again find a $\sqrt{\epsilon}$ dependence

$$D(\epsilon) \sim I_0 - B\sqrt{\epsilon},$$

with

$$B=(\pi^2/3)\ln(m_\mu/m_e) \sim 17.5. \tag{14}$$

The convergence is now dramatically improved and the results, shown in Fig. 4, confirm the $\sqrt{\epsilon}$ behavior *and the slope* $-B$. The Padé extrapolation is

$$I_0=21.20 \pm 0.15. \tag{15}$$

We also evaluated $D(0)$ directly and obtained

$$D(0)=I_0=21.3 \pm 0.3. \tag{16}$$

The independently determined results given in Eq. (6), (8), (11), (12), (15), and (16) are beautifully consistent. We combine Eqs. (6) and (11) to obtain our final result:

$$\Delta a_{\text{ph-ph}}=(21.32 \pm 0.05)(\alpha/\pi)^3. \tag{17}$$

With use of the new value[12] computed without quantum electrodynamics,

$$\alpha^{-1}=137.035\,987(29),$$

and[13]

$$a_e^{(6)}=(1.195 \pm 0.026)(\alpha/\pi)^3,$$

as well as the estimated eighth-order correction[14]

$$a_\mu^{(8)}=(150 \pm 70)(\alpha/\pi)^4=(4 \pm 2)\times 10^{-9},$$

Eq. (17) implies[15]

$$a_\mu^{\text{QED}}=(1165\,852 \pm 2)\times 10^{-9}. \tag{18}$$

Including the latest evaluation of the hadronic contribution,[16]

$$a_{\text{had}}=(66 \pm 10)\times 10^{-9}, \tag{19}$$

we obtain for the theoretical value

$$a_\mu^{\text{th}}=(1165\,918 \pm 10)\times 10^{-9}. \tag{20}$$

This is to be compared with the latest value from the CERN $g-2$ experiment,[5]

$$a_{\text{exp}}=(1165\,895 \pm 27)\times 10^{-9}. \tag{21}$$

One of us (M.A.S.) would like to thank S. D. Drell for the kind hospitality extended at Stanford Linear Accelerator Center as well as the ample computer time made available. He is also grateful to S. Brodsky for encouragement and helpful discussions.

*Work supported by the U. S. Energy Research and Development Administration.

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[7]The standard deviation of results obtained in a classical Monte Carlo computation is given by $\{[V\int f^2\,dz - (\int f\,dz)^2]/N\}^{1/2}$, where V is the volume of the region and N is the number of points used. The contribution to σ is large when the singularity is closely probed and hence, the contribution to the integral from this region is systematically de-emphasized in the cumulative result.

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[16]V. Barger, W. F. Long, and M. G. Olsson, unpublished. See also F. Combley, in Proceedings of the International Symposium on Lepton and Photon Interactions at High Energies, Stanford, California, 1975 (to be published). There is also a weak-interaction contribution of $+2\times10^{-9}$.

Atomic Structure Effects in Negative Meson Capture*

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Correlations are noted in the variation with atomic number of kaonic, pionic, and muonic x-ray yields, and positron-annihilation lifetimes in annealed metals. These variations are believed to be due to the variation with Z of the electron density in the outer part of the target atoms.

The extensive data of Wiegand and Godfrey[1] on kaonic-x-ray absolute yields exhibit a striking variation with atomic number. In the present note (i) we point out the existence of correlated variations in related data from muonic[2-4] and pionic[5] atoms,[6] and in recent data on positron annihilation in annealed metals[7]; (ii) we comment on a previous explanation of the kaonic-yield variations[8]; and (iii) we put forth a different explanation of these correlated variations, one which, in fact, is strongly supported by recent theoretical work on negative meson capture.[9]

In Fig. 1(a) are plotted the measured absolute yields of several x-ray transitions (from 6→5 up to 11→10) from kaonic atoms[1]; the higher and lower transitions for a given element, those