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# Parametric Instabilities in Strongly Relativistic, Plane-Polarized Electromagnetic Waves*

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(Received 25 August 1975)

We investigate the stability of long-wavelength, plane-polarized electromagnetic waves when the oscillatory energy of the electrons exceeds their rest energy. The strong electron-mass variation destabilizes electron modes polarized along the electric field of the pump $\vec{E}_0$. The electron and ion modes decouple in the presence of such an intense pump and hence the ions do not strongly influence the instability.

The relativistic-mass oscillation in the presence of a large-amplitude, long-wavelength electromagnetic wave can strongly enhance the anomalous absorption of the wave in plasma. Tsintsadze[1] has shown that this mass variation can parametrically excite plasma waves. More recently, Drake *et al.*[2] have shown that, because of the electron-mass changes, electrostatic perturbations of long-wavelength electromagnetic waves "slow down" in regions of high intensity, steepen, and then break.

In this paper we investigate the stability of long-wavelength, plane-polarized electromagnetic waves when the oscillatory energy of the electrons exceeds their rest energy. The electron velocity oscillates periodically between $\pm c$. We show (1) that the wave is unstable to pure electron perturbations polarized along $\vec{E}_0$, and (2) that the decay into coupled electron and ion modes, which occurs in the nonrelativistic limit,[3] does *not* take place. This result is in sharp contrast with the calculations of Max and Perkins[4] for a circularly polarized pump where the ions play an important role.

The novelty of the present calculation is associated with the treatment of the highly anharmonic mass variation of the electrons as their velocity oscillates between $\pm c$. Rather than expand the time dependence of the mass in an infinite Fourier series, we assume that the electrons are so massive that they only respond to the perturbed fields as their velocity passes through zero in crossing between $\pm c$. This approximation is valid to lowest order in an expansion in the small parameter $(1+P_m^2)^{-1/2}$, $P_m$ being the amplitude of the electron momentum (normalized to $m_e c$) in response to $\vec{E}_0$.

The electron motion follows from the relativistic kinetic equation in one dimension,

$$\{\partial/\partial t+[p/(1+p^2)^{1/2}]\,\partial/\partial x-[\tilde{E}(x,t)+E_0(t)]\,\partial/\partial p\}f(x,p,t)=0, \tag{1}$$

where we have normalized the electric field, momentum, distance, and time to $m_e\omega_p c/|e|$, $m_e c$, $c/\omega_p$, and $\omega_p^{-1}$, respectively, $\omega_p$ and $m_e$ being the electron plasma frequency and rest mass. With neglect of the perturbed field $\tilde{E}$, $f$ is a function of $q=p-p_0(t)$, where $p_0(t)=\int_0^t d\tau E_0(\tau)$ is the oscillatory momentum of the electron in $E_0$. The self-consistent time dependence of $p_0(t)$ was investigated by Akhiezer and Polovin[5] (see also Ref. 2 for a simpler discussion), who found that $p_0(t)$ is periodic over the time interval $\tau_0=2\pi/\omega_0=4(2P_m)^{1/2}$ when $P_m\gg 1$. We will investigate the stability of small perturbations around $p_0(t)$. It is convenient to change variables from $p$ to $q$ in (1). We then linearize the resultant equation and solve for the $k$th Fourier component of the perturbed distribution $\tilde{f}(x, q, t) = f(x, p, t) - f_0(q)$:

$$\tilde{f}_k = \int_0^t d\tau\, \tilde{E}_k(\tau) \exp[ikX(t, \tau, q)]\, \partial f_0/\partial q\,, \tag{2}$$

where

$$X(t, \tau, q) = \int_\tau^t d\tau_1\, (q + p_0)/[1 + (q + p_0)^2]^{1/2} \tag{3}$$

is the trajectory of a particle in $E_0$. When the thermal velocities of the electrons are nonrelativistic, $q \ll 1$ and (2) takes the form

$$\tilde{f}_k = \int_0^t d\tau\, \tilde{E}_k(\tau) \exp[ikX_0(t, \tau) - ikqh(t, \tau)]\, \partial f_0/\partial q\,, \tag{4}$$

where $X_0(t, \tau) = X(t, \tau, 0)$ and

$$h(t, \tau) = \int_\tau^t d\tau_1\, (1 + p_0^2)^{-3/2}. \tag{5}$$

In the nonrelativistic limit $h(t, \tau) = t - \tau$ and $qh(t, \tau)$ represents the free streaming of the electrons. In the present calculation, as the electron velocity approaches $c$, the spread of electron velocities in the lab frame approaches zero, since the instantaneous effective thermal velocity of a particle is $q/(1 + p_0^2)^{3/2}$. From the point of view of $\tilde{E}$, the electrons are periodically cooled. To lowest order in $P_m^{-1}$, we can treat the thermal velocity of an electron as zero at all times except the short intervals during which $p_0(t) \approx 0$. As a function of $\tau$, $h(t, \tau)$ is therefore a "stair step" function which is a constant unless $p_0(\tau) \simeq 0$ when $h(t, \tau)$ abruptly changes by

$$\Delta = \int_0^{\tau_0/2} d\tau_1\, [1 + p_0^2(\tau_1)]^{-3/2} \simeq (2/P_m)^{1/2}. \tag{6}$$

We conveniently divide time into half-periods $\tau_0/2$ of the pump such that $h$ is a constant over each half-period. If $t$ is in the $n$th half-period and $\tau$ in the $m$th, $h(t, \tau) \simeq (n - m)\Delta$.

We now insert $\tilde{f}_k$ in (4) into the first-order Poisson equation and carry out the $q$ integration for a Maxwellian velocity distribution. After including the cold-ion response in the equation, we find the following integral equation for $\tilde{E}_k(t)$:

$$\tilde{E}_k(t) = -\int_{-\infty}^t d\tau\, h(t, \tau)\tilde{E}_k(\tau) \exp[-k^2a^2h^2(t, \tau)/4 + ikX_0(t, \tau)] - \delta^2 \int_{-\infty}^t d\tau\, (t - \tau)\tilde{E}_k(\tau)\,, \tag{7}$$

where $\delta = (m_e/m_i)^{1/2}$. Equation (7) is valid for arbitrary pump strength $|E_0|$ and it therefore correctly describes the usual decay of $E_0(t)$ into plasma waves and ion fluctuations[3,6] as well as electrostatic instabilities arising from weak electron-mass oscillations.[1] Since these instabilities have been thoroughly analyzed by other authors, we will confine our discussion to the strongly relativistic pump.

To simplify the analysis, we ignore the ion response and investigate the stability of electron modes. This approximation will be justified *a posteriori* when the ion motion is included. With transformation of $\tilde{E}_k(t)$ to the oscillation frame of the electrons by defining $\tilde{A}_k(t) \equiv \tilde{E}_k \exp[-ikX_0(t, 0)]$, (7) simplifies to

$$\tilde{A}_k(t) = -\int_{-\infty}^t d\tau\, h(t, \tau)\tilde{A}_k(\tau) \exp[-k^2a^2h^2(t, \tau)/4]. \tag{8}$$

$h(t, \tau)$ is a constant over each half-period $\tau_0/2$ so that it can be factored out of the $\tau$ integral over this interval as follows:

$$\tilde{A}_k(t) = -\sum_{m=-\infty}^{n} (n - m)\Delta \exp\{-[ka\Delta(n - m)/2]^2\} \int_{(m-1)\tau_0/2}^{m\tau_0/2} d\tau\, \tilde{A}_k(\tau)\,; \tag{9}$$

$t$ again lies in the $n$th half-period of the pump. $\tilde{A}_k(t) \equiv \tilde{A}_k^{\,n}$ is a constant as long as $t$ remains within the $n$th interval so that the integral in (9) can be carried out. Equation (9) then reduces to

$$\tilde{A}_k^{\,n} = -4\sum_{m=0}^{\infty} m\tilde{A}_k^{\,n-m} \exp[-2(k\lambda_D^{\,0}m)^2]. \tag{10}$$

Because of the effective cooling of the electrons, their Debye shielding length reduces to $\lambda_D^{\,0} = a(4P_m)^{-1/2}$. We define the complex phase shift of $\tilde{A}_k(t)$ during $\tau_0$ as $\omega\tau_0$ so that $\tilde{A}_k^{\,n} \propto \exp(in\omega\tau_0/2)$. The dispersion relation $\epsilon_e(\omega, k)$ then follows from (10),

$$\epsilon_e(\omega, k) = 1 + 4\sum_{m=0}^{\infty} m \exp(i\omega\tau_0 m/2 - 2k^2\lambda_D^{\,02}m^2) \equiv 1 + \chi_e(\omega\tau_0, k\lambda_D^{\,0}) = 0\,. \tag{11}$$

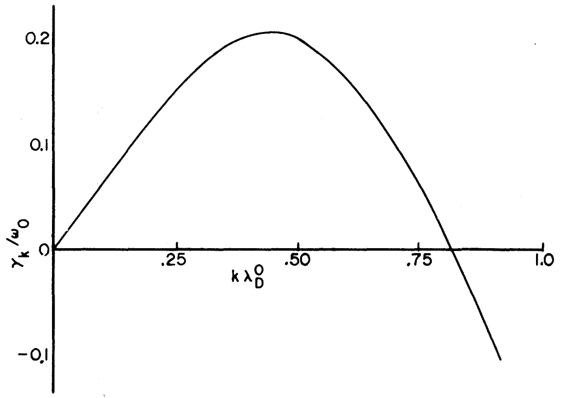


FIG. 1. The growth rate $\gamma_k$ of the electron instability as a function of $k$. The maximum growth rate is $0.21\omega_0$ at $k\lambda_D{}^0 \approx 0.44$.

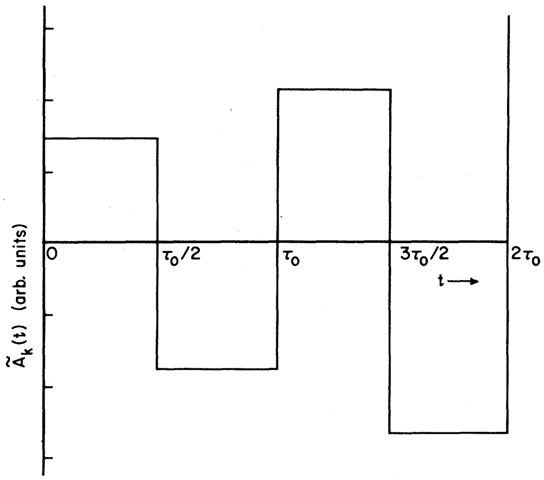


FIG. 2. The time dependence of $\tilde{A}_k(t)$, the electric field of the unstable mode in the electron oscillating frame, for $\gamma_k = 0.2\omega_0$. The phase of $\tilde{A}_k$ is periodic over $\tau_0 = 2\pi/\omega_0$, the period of the pump, and contains all harmonics of the pump fundamental frequency $\omega_0$.

The discrete summation in $\chi_e$ is analogous to the full $Z$ function of linear theory so that $\epsilon_e(\omega, r)$ includes electron damping. We analytically solve (11) for $\omega$ in two limits as follows:

$$\omega/\omega_0 = \begin{cases} 1 + i2k\lambda_D{}^0/\pi & \text{if } (k\lambda_D{}^0)^2 \ll 1\,, \\ 1 + i(\ln 4 - 2k^2\lambda_D{}^{02})/\pi & \text{if } (k\lambda_D{}^0)^2 \gtrsim 1\,. \end{cases} \qquad (12)$$

We show in Fig. 1 the results of the numerical solution of (11) for $\gamma_k = \mathrm{Im}\,\omega$. In the absence of thermal corrections, the perturbation is stable as discussed in Ref. 2. Note also that because of the effective electron cooling at velocities near $c$, short-wavelength modes which would be heavily damped in the absence of $E_0$ are actually unstable. $\omega$ does not, of course, represent the standard frequency of a linear wave. The actual time dependence of $\tilde{A}_k(t)$ is shown in Fig. 2. The phase is periodic over the time interval $\tau_0$ as in the calculation of Tsintsadze for the weakly relativistic pump. The electrons are so massive while they are moving at roughly $\pm c$ that they do not respond to $\tilde{A}_k$ and hence $\tilde{A}_k$ does not evolve during these intervals. As the electrons pass through zero velocity, their mass reduces to $m_e$ and they rapidly respond to $\tilde{A}_k$, changing its phase by $\pi$.

The same essential approximations which we have just made in solving for $\epsilon_e(\omega, k)$ enable us to include the ion response. We again assume that $h(t, \tau)$ in the first term on the right-hand side of (7) is a constant over each interval $\tau_0/2$. This term is then a functional of $\tilde{E}_k$ only through

$$\tilde{B}_k{}^n = \int_{(n-1)\tau_0/w}^{n\tau_0/2} d\tau\, \tilde{E}_k(\tau) \exp[-ikX_0(\tau, 0)]\,.$$

Solving (7) for $\tilde{E}_k(t)$ as a functional of $\tilde{B}_k{}^n$, multiplying the result by $\exp[-ikX_0(t,0)]$, and integrating over a half-period, we obtain the following equation for $\tilde{B}_k{}^n$ alone:

$$\tilde{B}_k{}^n = -4 \sum_{m=0}^{\infty} m\tilde{B}_k{}^{n-m} \exp(-2k^2\lambda_D{}^{02}m^2) + 4\delta \int_{(n-1)\tau_0/2}^{n\tau_0/2} d\tau \int_{-\infty}^{\tau} d\tau_1 \sin[\delta(\tau - \tau_1)]$$
$$\times \exp[-ikX_0(\tau, \tau_1)] \sum_{m\,p} m\tilde{B}_k{}^{p-m} \exp(-2k^2\lambda_D{}^{02}m^2)\, \theta(p\tau_0/2 - \tau_1)\, \theta(\tau_1 - (p-1)\tau_0/2)\,. \qquad (13)$$

The ion response enters through the second term on the right-hand side of (13). We can carry out the time integrals in (13) if we approximate the electron velocity by $\pm c$ and assume $\delta\tau_0 < 1$. The dispersion relation then follows once we assume $\tilde{B}_k{}^n \propto \exp(-in\omega\tau_0/2)$:

$$\epsilon(\omega, k) = 1 + \chi_e(\omega\tau_0, k\lambda_D{}^0)\{1 + [\sin(\tfrac{1}{4}k\tau_0)/\tfrac{1}{4}k\tau_0]^2 (\tfrac{1}{2}\delta\tau_0)^2/[4\sin^2(\tfrac{1}{4}\omega\tau_0) - (\tfrac{1}{2}\delta\tau_0)^2]\}\,. \qquad (14)$$

Since $\delta\tau_0 \ll 1$, the ion response is negligible unless $\omega \sim \delta \ll 1$, justifying our previous neglect of the ion motion for the electron instability. An important result of this paper is that the ion and electron modes decouple when $P_m \gg 1$. For low-frequency waves $\chi_e(\omega\tau_0, k\lambda_D{}^0) \simeq \chi_e(0, k\lambda_D{}^0)$, which can be approximated as

$$\chi_e(0, k\lambda_D{}^0) \simeq \begin{cases} (k\lambda_D{}^0)^{-2} \text{ when } (k\lambda_D{}^0)^2 \ll 1, \\ 4\exp(-2k^2\lambda_D{}^{02}) \text{ when } (k\lambda_D{}^0)^2 \gtrsim 1. \end{cases} \tag{15}$$

We now solve for $\omega$ explicitly as follows:

$$\omega = \pm\delta\{1 - [\sin(\tfrac{1}{4}k\tau_0)/\tfrac{1}{4}k\tau_0]^2/[1 + \chi_e{}^{-1}(0, k\lambda_D{}^0)]\}^{1/2}. \tag{16}$$

The electron response appears in the second term on the right-hand side of (16) and is always less than unity so that the low-frequency mode is stable. To understand our expression for $\omega$ in (16), we consider an ion density perturbation with a wave vector $k$. The electrons oscillate in response to $E_0$ with a high-frequency displacement $\Delta x \sim \tau_0/4$ and thereby experience a periodic phase shift $k\tau_0/4$ of the low-frequency wave. The average low-frequency field seen by the electrons (averaged over the high-frequency displacement) is reduced by $[\sin(\frac{1}{4}k\tau_0)/\frac{1}{4}k\tau_0]^2$ relative to the field at a fixed spatial point. This reduction factor appears in (16). When $k\tau_0/4 \gtrsim \pi$, the electron response can be neglected and the ions oscillate at their natural frequency $\delta$. For long-wavelength modes $\omega$ approximately equals $\pm\delta k\lambda_D{}^0$, the usual ion acoustic frequency reduced by the effective cooling of the electrons.

Because of the decoupling of the electron and ion modes in the strongly relativistic limit, the nonlinear development of the parametric instabilities in this regime will be strikingly different from the corresponding development of the electron-ion modes of the nonrelativistic pump. The ion modes will probably not play an important role in the nonlinear evolution of instabilities of the strong pump, so that plasmon trapping by localized density cavities, which occurs in the nonlinear state of the instabilities of the nonrelativistic pump, will not take place.

The instabilities investigated in this paper may have a strong influence on a recent theory[7] of cosmic-ray acceleration by pulsar fields. Kennel, Schmidt, and Wilcox propose that the wave energy of super-relativistic plasma waves propagating outward from pulsars is converted into directed particle motion thus producing large numbers of high-velocity particles. In the reference frame of the high-velocity particles, their relativistic plasma wave reduces to our homogeneous pump.[8] Hence, the instabilities discussed in this paper may break up the relativistic plasma wave before energy transfer to directed particle motion can occur.

*Work supported in part by National Science Foundation Grant No. GP 40983-PJ69 and by U. S. Atomic Energy Commission–Energy Research and Development Administration Contract No. AT(04-3)-34 PA157.

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