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Extraction of Resonance Parameters from Nuclear Scattering Data*

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A procedure is described for extracting the pole position of a resonant amplitude from a knowledge of the phase shift versus energy. Pole positions are determined for the four p-wave resonances in the nucleon-α system with high precision. The use of the pole position to define the mass of unstable nuclear states is discussed.

A recent nuclear-energy-level compilation[1] indicates rather large uncertainties in the quoted masses and widths of the $p_{3/2}$ and $p_{1/2}$ resonances in ^{5}He and ^{5}Li (see Table I). It is customary to define the masses of these states in terms of the observed central energy of an experimental resonance peak and in the $p_{1/2}$ cases it is not surprising that large uncertainties are present, since the corresponding N-α phase shifts do not pass through $\pi/2$ and no sharp group appears in the experimental spectra. It could be argued that although the position of a prominent peak may be useful for characterizing a resonance experimentally, this position does not necessarily have a compelling theoretical significance as the definition of the mass of the level.

From the viewpoint of the analytic properties of the scattering amplitude, a resonance is characterized by its pole position and residue, and the pole position could serve as a suitable definition of the mass of an unstable level. In this paper we will study the utility of this alternative definition by extracting the pole positions of the N-α resonances from published energy-dependent phase-shift analyses. Since the pole occurs at an energy obviously not accessible to experimentation, our procedure involves a continuation of measured quantities to complex energies, but as we will see in the cases studied, the pole positions can be determined with much greater precision than that quoted in Table I. Similar results have been obtained in the π-N system[2] where the pole position of the 3-3 resonance has been accurately determined by a variety of methods.

TABLE I. Resonance parameters from Ref. 1.

Level	$E_{c.m.}$ (MeV)	$\Gamma_{c.m.}$ (MeV)
$n\alpha, p_{3/2}$	0.89 ± 0.05	0.30 ± 0.01
$n\alpha, p_{1/2}$	4.9 ± 1.5	2.0 ± 0.8
$p\alpha, p_{3/2}$	1.97 ± 0.05	≈ 0.8
$p\alpha, p_{1/2}$	7–12	2.5 ± 1.0

We first treat n-α scattering, adapting a method developed by Ball and Goble[3] to the chargeless case. We write the p-wave amplitude for either $J=\frac{3}{2}$ or $\frac{1}{2}$ in the usual way,

$$f(k)=\exp(i\delta)\sin\delta/k=k^2/(k^3\cot\delta-ik^3), \qquad (1)$$

where k is the center-of-mass wave number and δ is the phase shift. Since $k^3\cot\delta$ is analytic[4] at $k=0$, the usual effective-range expansion can be

carried out,

$$K(k) = k^3 \cot\delta = \sum_{n=0}^{N} a_n k^{2n}, \tag{2}$$

and we will assume that such an energy-dependent fitting has been performed and the coefficients a_n determined from experiment. The denominator of (1) then becomes the polynomial

$$D(k) = \sum_{n=0}^{N} a_n k^{2n} - ik^3, \tag{3}$$

and a resonance of the system would manifest itself as a pole of $f(k)$ or a zero[5] of $D(k)$ in the fourth quadrant of the complex k plane. The complex center-of-mass energy of the resonance is then $\hbar^2 k^2/2m$, where k is the resonant root of (3). Our result for the pole position is model independent in the sense that no resonance formula has been invoked to represent the amplitude. The error in the pole position may also be determined if the error matrix M_{nm} is available from the fitting of the a_n's. If $k = r$ is the resonant wave number such that $D(r) = 0$, then since r depends only on the coefficients a_n, its uncertainty Δr may be expressed as[6]

$$(\Delta r)^2 = \sum_{n,m=0}^{N} \frac{\partial r}{\partial a_n} M_{nm} \frac{\partial r}{\partial a_m}, \tag{4}$$

and since $\partial r/\partial a_n = -r^{2n}/D'(r)$, the error becomes

$$(\Delta r)^2 = [D'(r)]^{-2} \sum_{n,m} r^{2n} M_{nm} r^{2m}. \tag{5}$$

We thereby obtain, from the phase-shift analysis, an error estimate for the real and imaginary part of r.

The above procedure has been used to determine the $p_{3/2}$ and $p_{1/2}$ pole positions in ^{5}He from the energy-dependent phase-shift analysis carried out by Arndt and Roper[7] on the basis of 1469 pieces of n-α data between 0 and 21 MeV. It is particularly important in the case of broad resonances such as the $p_{1/2}$ to have data over a wide energy range in order to carry out an accurate continuation. This phase-shift analysis employed terms up to $N = 3$ and the errors in the pole positions have been computed from the published error matrices.[7] We find in our analysis that most of the zeros of (3) are statistical in character and change their position drastically if the order N is varied. The zero that we associate with the resonance, however, is very stable under changes in N. Table II lists the resulting center-of-mass pole positions and uncertainties and it can be seen that the results are much more precise than the corresponding values of Table I.

TABLE II. n-α pole positions.

Level	$\mathrm{Re}E_{\mathrm{c.m.}}$ (MeV)	$-\mathrm{Im}E_{\mathrm{c.m.}}$ (MeV)
$p_{3/2}$	0.7778 ± 0.0002	0.3196 ± 0.0006
$p_{1/2}$	1.999 ± 0.011	2.267 ± 0.008

The p-α case is complicated by the appearance of an essential singularity in the amplitude due to the Coulomb force. Several authors[3,8] have discussed the necessary modifications of the effective-range expansion. We work with the following function that is regular at $k = 0$,

$$K_C(k) = k^3 C_1^{\,2} \cot\delta + Q(k) = \sum_{n=0}^{N} a_n k^{2n}, \tag{6}$$

where

$$\begin{aligned} C_1^{\,2} &= 2\pi\eta(1+\eta^2)/[\exp(2\pi\eta) - 1], \\ \eta &= Z_1 Z_2 e^2 m/\hbar^2 k, \\ Q(k) &= \eta k^3 [\psi(i\eta) + \psi(-i\eta) - 2\ln\eta](1+\eta^2). \end{aligned} \tag{7}$$

m is the reduced mass, ψ is the digamma function, and δ is the nuclear phase shift in the presence of the point Coulomb interaction. The corresponding denominator function is

$$D_C(k) = \sum_{n=0}^{N} a_n k^{2n} - Q(k) - iC_1^{\,2} k^3 \tag{8}$$

and, with the a_n determined from experiment, our problem reduces to searching[9] the fourth quadrant of the k-plane for points at which $D_C(k) = 0$. The error in the location of the zero follows from the error matrix as in the n-α case. We have used a 0–23-MeV phase-shift analysis[10] based on 1343 pieces of p-α data to determine the pole positions of the $p_{3/2}$ and $p_{1/2}$ states of ^{5}Li. The results are listed in Table III.

All errors quoted up to this point have included only those implied by the phase-shift error matrices. There could be other sources of error such as biases involved in using the effective-range expansion to represent the amplitude or errors of extrapolation. The experimental energy scale is probably not known to the uncertainty quoted in the ^{5}He $p_{3/2}$ case. We feel that a more conservative uncertainty of at least 5 keV should be associated with all results.

A comparison of our results with those of Table I indicates that the real parts of the pole positions are always less than the accepted posi-

TABLE III. p-α pole positions.

Level	$\mathrm{Re}E_{\mathrm{c.m.}}$ (MeV)	$-\mathrm{Im}E_{\mathrm{c.m.}}$ (MeV)
$p_{3/2}$	1.637 ± 0.005	0.646 ± 0.003
$p_{1/2}$	2.858 ± 0.014	3.041 ± 0.023

tions of the resonances. If the $p_{3/2}$-$p_{1/2}$ splitting is defined as the difference between the corresponding real parts, then we also find that this splitting is remarkably equal in the ^{5}He and ^{5}Li cases, amounting to 1.22 ± 0.01 MeV.

It is often found in fitting resonant nuclear data with several possible forms of R-matrix theory that the emergent energy and width parameters vary over an uncomfortably wide range.[11] Typically such resonance formulas have a momentum dependence introduced into the width and thus the pole position and the resonant-energy parameters of the formula need not coincide. In the π-N problem[12] it has been found that data fitted equally well with several resonance formulas yield virtually the same pole position even though the position and width parameters in the formulas vary considerably. It would be interesting to see if this effect is also present in the nuclear cases. In the Humblet-Rosenfeld resonance formalism, the pole position is a fundamental parameter of the theory and Kraus and Linck[13] have made such fits to n-α and p-α data that yield resonance parameters closer to our results than do corresponding R-matrix fits.

In conclusion we urge that the pole positions of all nuclear resonances be determined whenever possible.

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