suppresses saturation of the amplification when $(V_1 - V_0)(V_2 - V_0) < 0$. Moreover, if $V_1 V_2 < 0$, it has the same effect as a finite extent of the interaction region in the coherent case and absolute instabilities are restored.

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Breaking of Large-Amplitude Waves as a Result of Relativistic Electron-Mass Variation*

J. F. Drake, Y. C. Lee, K. Nishikawa, and N. L. Tsintsadze†

Department of Physics, University of California, Los Angeles, California 90024

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The role of the relativistic electron-mass variation in the evolution of large-amplitude electromagnetic waves near cutoff and in the generation of plasma waves by linear mode conversion is investigated. Shock formation and subsequent wave breaking along the electric field of the wave will enhance the absorption of the wave by the plasma.

The anomalous absorption of a large-amplitude electromagnetic wave near its cutoff has been the subject of intense research by physicists over the last decade. Most of this research has centered on the generation of coupled electron plasma and ion modes and their subsequent absorption by the plasma.[1] More recently, the role of the electron-mass variation in the absorption of large-amplitude waves has been recognized. Tsintsadze[2] has shown that the electron-mass oscillation in a large-amplitude wave can parametrically amplify plasma waves and that the growth rates of the resulting instabilities exceed those of the decay instability.

In the present paper we investigate the nonlinear evolution of large-amplitude waves near cutoff including the relativistic mass changes of the electrons but neglecting ion motion. Initial spatial irregularities of the wave parallel to its electric field E steepen, form shocks, and then break. Electrons are heavier in the region of the highest wave intensity so the wave "slows down" in this region, allowing the lower-intensity portion of the wave to "catch up."

Although we cannot analytically follow the evolution of the wave after it has broken, wave breaking observed in computer simulations invariably causes strong plasma heating so this mechanism may have an important influence on the absorption of large-amplitude waves in plasma. Since the waves break in the absence of ion participation, substantial absorption of the wave may occur before the development of the decay instability or related instabilities, which cause anomalous absorption of the wave over ion time scales. Indeed, to neglect the ion dynamics, which play a crucial role in these later instabilities and in profile modification by the large-amplitude wave in the presence of a density gradient, we must assume that wave breaking takes place before these other processes significantly develop.

We begin with the cold-fluid equations for the electrons and assume that the ions form a uniform neutralizing background:

$$dp/dt = -E(x,t), \tag{1a}$$

$$\partial n/\partial t + \partial nv/\partial x = 0, \tag{1b}$$

$$\partial E/\partial x = -(n_e - 1), \tag{1c}$$

where $d/dt = \partial/\partial t + v\,\partial/\partial x$, $v(x,t) = p/(1+p^2)^{1/2}$, and we have normalized all times, velocities, momenta, and distances to ω_p^{-1}, c, m_0c, and c/ω_p, respectively. The magnetic induction has been neglected for an electromagnetic wave near cutoff or for an electrostatic wave. Operating on (1a) with d/dt, we find

$$d^2p/dt^2 + p(1+p^2)^{-1/2} = 0. \tag{2}$$

The factor $(1+p^2)^{-1/2}$ represents the local decrease of the plasma frequency due to the relativistic mass increase of the electrons. We now transform to the frame x_0 moving with the fluid, defined by $x_0 = x - \int_0^t dt\, v(x_0, t)$, where x is the location of the fluid in the lab frame. In the moving frame

$$\partial^2 p(x_0, t)/\partial t^2 + p(1+p^2)^{-1/2} = 0, \tag{3}$$

a generalization of the equation describing the oscillation of a large-amplitude plasma wave.[3] Using (3), we find $x_0(x,t) = x + \partial p(x_0,t)/\partial t$, where we have chosen $\partial p(x_0, t=0)/\partial t = 0$ for simplicity. The first integral of (3) yields

$$(\partial p/\partial t)^2/2 + (1+p^2)^{1/2} = H(x_0) = [1 + P_m^2(x_0)]^{1/2}. \tag{4}$$

The first and second terms on the left-hand side of (4) are the wave energy and particle energy, respectively, so (4) expresses energy conservation, P_m being the maximum momentum. The time evolution of p is analogous to that of a particle moving in a potential $(1+p^2)^{1/2}$. The motion is periodic with frequency[4]

$$\omega(H) = \pi\{2^{1/2}\int_0^{P_m} dp/[H - (1+p^2)^{1/2}]^{1/2}\}^{-1}, \tag{5a}$$

$$= 1 - 3P_m^2/16 \text{ for } P_m \ll 1, \tag{5b}$$

$$= \pi/(8P_m)^{1/2} \text{ for } P_m \gg 1, \tag{5c}$$

which is a monotonically decreasing function of H.[5] Since H is an arbitrary function of x_0, each particle oscillates independently in this frame at its appropriate frequency.

We now consider a wave which is initially spatially uniform (H is independent of x_0) and perturb it at $t=0$ by modulating its amplitude (H is now a weak function of x_0). The particles initially oscillate in phase, but since their periods are slightly different, the relative phase of the individual particles increases secularly. Because of the phase shift, some of the particles begin to catch up with their neighbors, thus increasing the local density. This process is illustrated in Fig. 1 where we show the results of a numerical calculation of the particle displacement x as a function of time for $p(x,0) = 3 + 0.6\sin(\pi x/5)$. The motion of each individual particle is periodic but the electron density (line density for a given t) evolves rapidly as the relative phase of the particles increases. In Fig. 2 the electron density is shown as a function of x at several times around $t \sim 14$. The crucial difference between the evolution of the wave in Figs. 1 and 2 and that of the nonrelativistic large-amplitude plasma wave is that the plasma wave is *periodic* over the interval $2\pi/\omega_p$ while in the present calculation the phase shift of the particle is secular and hence the peak of the density profile *increases* with each period until the wave breaks. A half period after the curves shown in Fig. 2, wave breaking occurs.

The electron density in the lab frame is given

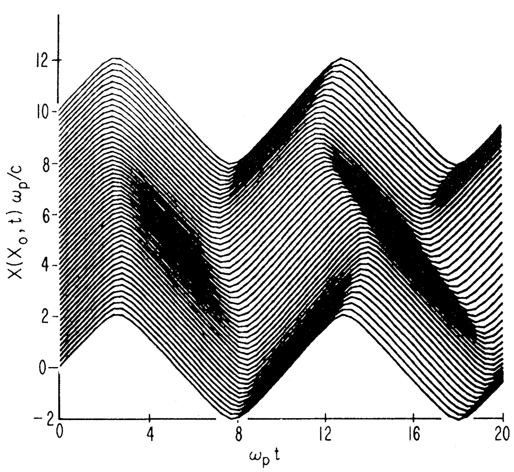

FIG. 1. The particle displacement $x(x_0,t)$ is shown as a function of time. The particles are initially distributed uniformly but have momentum $p(x,0) = 3 + 0.6 \times \sin(\pi x/5)$ with $\partial p(x, t=0)/\partial t = 0$.

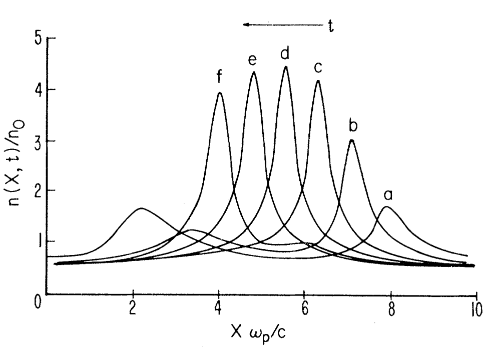

FIG. 2. The spatial dependence of the electron density is shown at $\omega_p t = 12.4$, 13.2, 14.0, 14.8, 15.6, and 16.4.

by $(\partial x/\partial x_0)^{-1}$ so we calculate the time t_s when the wave breaks by requiring that

$$(\partial x/\partial x_0)=0=1-\partial^2 p(x_0,t)/\partial x_0\partial t. \qquad (6)$$

We conveniently represent $p(x_0,t)$ as $h(H(x_0), \varphi(x_0,t))$, where $\varphi(x_0,t)=\omega(H(x_0))t$. $h(H,\varphi)$ is periodic in φ over the interval 2π and the x_0 dependence of p enters only through the energy $H(x_0)$. Equation (6) takes the form

$$1=(\omega h_{\varphi\varphi}t_s\omega_H+\omega_H h_\varphi+\omega h_{\varphi H})H_{x_0}, \qquad (7)$$

where the subscripts φ, H, and x_0 represent partial derivatives with respect to that variable. The last two terms on the right-hand side of (7) are periodic, so if $t_s \gg 2\pi/\omega$, they can be neglected. Using (3),

$$1=-p(x_0,t_s)(1+p^2)^{-1/2}t_s\partial\ln\omega/\partial x_0. \qquad (8)$$

The wave will first break when $p(x_0,t_s)$ is a maximum so $p(x_0,t_s)$ can be replaced by $P_m(x_0)$ in (8). t_s will be a minimum where the period has the fastest spatial variation (phase shift is greatest):

$$t_s\simeq(1+P_m^2)^{1/2}P_m^{-1}(x_s)|\partial\ln\omega/\partial x_s|^{-1}, \qquad (9a)$$

$$\simeq\tfrac{8}{3}|P_m^2\partial P_m/\partial x_s|^{-1} \text{ for } P_m\ll 1, \qquad (9b)$$

$$\simeq 2P_m|\partial P_m/\partial x_s|^{-1} \text{ for } P_m\gg 1, \qquad (9c)$$

where $\partial^2\ln\omega/\partial x_s^2=0$ defines x_s. The shock formation time is a sensitive function of the initial modulation of the wave. Since in most cases of interest the wave may have to traverse a region of underdense plasma where modulation or filamentation instabilities can break up the wave perpendicular to its direction of propagation, the wave may have a substantial modulation when it reaches cutoff, thereby significantly reducing t_s.

We now consider the effect of the relativistic electron-mass variation on the generation of a large-amplitude plasma wave by an electromagnetic wave obliquely incident on a density gradient. We model this process by adding a source $\epsilon\cos\omega_0 t$ on the right-hand side of Eq. (3).[6,7] Assuming the electrons are only weakly relativistic and ignoring for the present any spatial dependence of ϵ or ω_p, we obtain

$$\partial^2 p(x,t)/\partial t^2+p(x,t)(1-p^2/2)=\epsilon\cos\omega_0 t. \qquad (10)$$

The normal modes of (10) when $\epsilon\sim 0$, $\omega_0\sim 1$, and the relativistic mass changes of the electrons can be neglected are $\cos\omega_0 t$ and $\sin\omega_0 t$. The nonlinear term will weakly couple these modes so we assume a solution of the form

$$p(t)=2(\epsilon/3)^{1/3}[f(t)\cos\omega_0 t-g(t)\sin\omega_0 t], \qquad (11)$$

where $f(t)$ and $g(t)$ are slowly varying functions of t. We insert this expression for $p(t)$ into (10), neglect second derivatives of f and g, and equate the coefficients of $\cos\omega_0 t$ and $\sin\omega_0 t$. We find

$$\partial f/\partial\tau=\partial H/\partial g,\quad \partial g/\partial\tau=-\partial H/\partial f, \qquad (12)$$

where $H=(f^2+g^2-\delta)^2/4+f$ and $t=\omega_0(\epsilon/3)^{1/3}\epsilon^{-1}\tau$, and $\delta=2(1-\omega_0^2)(\epsilon/3)^{1/3}\epsilon^{-1}$ represents the mismatch between the frequency of the source and that of the linear normal mode. f and g are canonical conjugates of the Hamiltonian H, which is a constant of the motion. The stationary points f_0, g_0 of (12) are obtained by requiring $\partial f/\partial\tau=\partial g/\partial\tau=0$:

$$(f_0^2-\delta)f_0+1=0,\quad g_0=0. \qquad (13)$$

Equation (13) has three real solutions when $\delta^3>\frac{27}{4}$ and only one when the inequality is reversed. The solutions f_0 are $\pm\delta^{1/2}$, δ^{-1} for large positive δ and δ^{-1} for large negative δ. The stationary points are shown in Figs. 3(a) and 3(b) where we

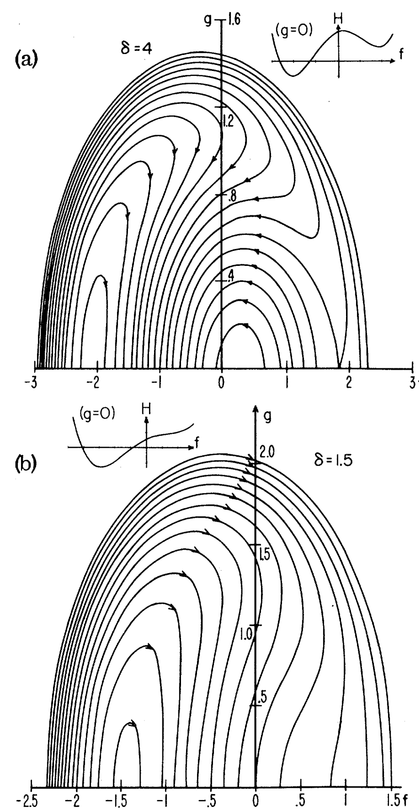

FIG. 3. Constant-H curves are plotted in the f,g plane for $\delta=4$ in (a) and for $\delta=1.5$ in (b). The energy increment between adjacent curves is constant in (a) and (b). Cross sections along $g=0$ showing H as a function of f are also displayed.

plot the constant-H curves in the f,g plane when $\delta^3 \lessgtr \frac{27}{4}$, respectively (the curves are symmetrical under the transformation $g \to -g$). The stationary points correspond to the well, the peak, and the saddle point (from left to right) in Fig. 3(a) and to the well in Fig. 3(b). Figure 3, of course, also illustrates the entire nonlinear evolution of f and g. H is conserved, so the time evolution of f and g for a given system is determined by moving along the appropriate constant-H curve in the direction indicated by the arrows. f and g are periodic over an interval $T(H)$. Moreover, since the amplitude of p is proportional to $(f^2+g^2)^{1/2}$, which varies substantially over the majority of constant-H curves in Fig. 3, the time evolution of p will be characterized by large, periodic, amplitude modulations.

The energy H will change as ϵ evolves in time and therefore the energy of a given system and consequently $p(t)$ will be a sensitive function of how we turn on ϵ. When ϵ is switched on suddenly, the particle will follow the constant-H curve passing through the origin, which circles the peak as in Fig. 3(a) when $\delta^3 > \frac{27}{2}$ and circles the well as in Fig. 3(b) when $\delta^3 < \frac{27}{2}$ (the transition occurs at $\delta^3 = \frac{27}{2}$ when the constant-H curve passing through the origin intersects the saddle point). p will exhibit large amplitude modulations. The rapid change in the topology of the constant-H curve followed by the individual particles in the vicinity of $\delta^3 \approx \frac{27}{2}$ will strongly influence the wave structure in a density gradient where $\delta \propto \omega_p^2(x) - \omega_0^2$. Equation (10) is valid as long as the displacement of the particles by the wave is small relative to the scale length of ϵ. We consider a system in which $\delta^3 - \frac{27}{2}$ is positive in one region and negative in another. The structure and frequency of the oscillations in the two regions will be entirely different. In our previous investigation of the free wave, we found that wave breaking first occurred where the variation of the wave frequency was a maximum. In the present case the frequency changes most rapidly and hence wave breaking will take place in the transition region where $\delta^3 \approx \frac{27}{2}$. In the nonrelativistic limit $\epsilon \ll 1$ so the wave will break where $\omega_p(x) \approx \omega_0$.[7]

We now consider the time evolution of a system in which $\delta > 0$ and ϵ increase adiabatically (slow compared to the period of oscillation) from nearly zero to some final value. $\delta \propto \epsilon^{-2/3}$ concurrently decreases in time from a large initial value. The action $J = \oint g df$ (g and f are canonical conjugates) is an approximate constant of the motion. The constant-H waves initially have the topology shown in Fig. 3(a) with the peak (at $f_0 = \delta^{-1}$) lying very near the origin. The particles encircle the peak on the constant-H curve passing through the origin. The peak moves towards higher f as ϵ increases, and since $J \propto \delta^{-2} \approx 0$, the particles must follow the peak. Hence, $p(t) \simeq 2(\epsilon/3)^{1/3}\delta^{-1} \times \cos\omega_0 t$. The plasma responds linearly because of the large mismatch between the plasma and driving frequencies. However, when $\delta^3 \lesssim \frac{27}{4}$, the peak abruptly vanishes as the mass change of the electrons becomes important. The particles then follow trajectories similar to those passing through $g = 0$ in the right-half-plane of Fig. 3(b). When δ is also x dependent, wave breaking will take place in the vicinity of $\delta^3 \approx \frac{27}{4}$ where the structure of the wave changes rapidly. In the nonrelativistic limit $\delta^3 \gg \frac{27}{4}$ so the plasma continues to respond linearly and wave breaking does not occur.[8]

We emphasize again that the spatial variation of the local frequency of the wave causes the wave breaking of both the free and mode-converted waves. The phase of the wave advances more rapidly in the high-frequency region so the electron fluid in this region overtakes the fluid in the low-frequency region, causing the wave to break. The increase of the electron mass in the high-intensity region of the wave produces the frequency variation in the free wave while a combination of this effect and the spatial dependence of the plasma frequency in a density gradient causes the frequency variation of the mode-converted wave. The frequency variation of the nonrelativistic, mode-converted wave is produced solely by the latter effect.[7]

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†On leave from Georgian Academy of Sciences, Tbilisi, U. S. S. R.

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Interference Effects in Transition Radiation Produced by Swift Ion Clusters*

N. R. Arista and T. E. Huber
Centro Atómico Bariloche, Bariloche, Argentina
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We analyze the conditions for observing interference effects in transition radiation at optical frequencies produced by ion clusters of considerably smaller size than the wavelength of the emitted radiation. We consider the possibility of using these effects in the study of the clusters' structure and alignment when they traverse solid films. Numerical results are obtained for H_2^+ incident on aluminum.

When charged particles pass through the interface of two media with different optical properties electromagnetic radiation is emitted. This "transition radiation" (TR) has been studied, from both the microscopic[1] and macroscopic[2] point of view. Experiments give good agreement with the theoretical calculations.[3]

Recently, the irradiation of solid films with molecular-ion beams has been the subject of much interesting work, and in particular studies of transmission,[4] alignment effects,[5,6] and energy losses[7] of molecular beams in channeling and bombardment in random directions.

In this work we wish to consider interference effects in TR in the optical range generated by nonrelativistic ion clusters of much smaller size than the wavelength λ of the emitted radiation. Furthermore, we wish to consider the possibility of using these effects in the study of the structure of the clusters.

Let us consider a cluster of particles moving with average velocity v_0 through the surface $z = 0$; the particles are normally incident to this surface and come from a nonmagnetic medium with dielectric constant $\epsilon(\omega)$ (for $z < 0$) into the vacuum (for $z > 0$). The approximation of considering the first medium as semi-infinite is justified if the thickness d of the foil is large enough so that the processes at the two surfaces may be considered to be independent of each other ($d \gg \lambda$).

The current density generated by the cluster is

$$\vec{j}(\vec{r}, t) = \hat{z}\sum_n v_n Z_n e\, \delta(\vec{r} - \vec{r}_n - \hat{z}v_n t), \tag{1}$$

where v_n is the velocity of the nth particle, $z_n e$ its charge, and $\vec{r}_n = (x_n, y_n, z_n)$ its position at time $t = 0$. The sum over n runs over all particles of the cluster. We will consider $v_n = v_0 + u_n$ with $|u_n| \ll v_0$.

Performing a Fourier transformation from the variables x, y, z, t to the more convenient k_x', k_y', z, ω we obtain

$$\vec{j}(\vec{k}', z, \omega) = \sum_n \vec{j}_n(\vec{k}', z, \omega), \tag{2}$$

where $\vec{k}' = (k_x', k_y')$, $\vec{\rho}_n = (x_n, y_n)$, and

$$\vec{j}_n(\vec{k}', z, \omega) = (2\pi)^{-3/2}\hat{z}Z_n e\, \exp\{-i[\vec{k}\cdot\vec{\rho}_n + (z_n - z)\omega/v_n]\}.$$

The driving Hertz vector[2] produced by this current in an infinite homogeneous medium of dielectric constant ϵ is

$$\vec{\Pi}(\vec{k}', z, \omega) = (4\pi i/\omega\epsilon)\sum_n (k'^2 - \epsilon\,\omega^2/c^2 + \omega^2/v_n^2)^{-1}\vec{j}_n(\vec{k}', z, \omega). \tag{3}$$

In order to match the fields at the surface $z = 0$ we must add secondary fields to the driving fields described by Eq. (3). These secondary fields are solutions of the homogeneous wave equation with coefficients determined by the boundary conditions and lead to transition radiation. In this way we obtain for the radiative part of the Hertz vector in the vacuum region for nonrelativistic velocities

$$\vec{\Pi}_{\text{rad}}(\vec{k}', z, \omega,) = \vec{\Pi}_{\text{rad}}{}^0(\vec{k}', z, \omega)\sum_n Z_n (v_n/v_0) \exp[-i(\vec{k}'\cdot\vec{\rho}_n + \omega z_n/v_n)], \tag{4}$$