should not allow oneself to be too discouraged simply because (17) did not agree with some experiments.

[9]It may be argued that if Dirac's quantization condition $e_m e = n/2$ is not satisfied then the field of a magnetic monopole of strength e_m cannot be taken as a realizable physical situation. See Ref. 1.

Finite Improvement Renormalizes the Energy-Momentum Tensor*

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The problem of the improvement term of the energy-momentum tensor $\theta_{\mu\nu}$ in φ^4 theory is reconsidered. A unique finite improvement coefficient is shown to renormalize $\theta_{\mu\nu}$. Dimensional regularization is used and the improvement coefficient depends only on the space-time dimension. Up to three-loop order but not beyond, the value suggested by conformal arguments works. But if use is allowed of 't Hooft's methods to sum the divergences, then this value does work.

The problem of renormalizing the energy-momentum tensor in φ^4 theory has often been considered in the past.[1-6] As I will show, improvements to the treatment can be made.

In terms of the bare field φ_0, the Lagrangian is

$$\mathcal{L} = \tfrac{1}{2}(\partial\varphi_0)^2 - \tfrac{1}{2}m_0{}^2\varphi_0{}^2 - \tfrac{1}{24}g_0\varphi_0{}^4 . \tag{1}$$

Then the canonical energy-momentum tensor is[7]

$$T_{\mu\nu} = \partial_\mu\varphi_0\partial_\nu\varphi_0 - g_{\mu\nu}\mathcal{L} . \tag{2}$$

To renormalize the connected Green's functions of $T_{\mu\nu}$, it is sufficient[1-4] to define an improved energy-momentum tensor

$$\theta_{\mu\nu} \equiv T_{\mu\nu} - H_0(\partial_\mu\partial_\nu - g_{\mu\nu}\Box)\varphi_0{}^2 , \tag{3}$$

and to choose the improvement coefficient H_0 appropriately. I ignore the renormalization needed of the vacuum expectation value of $\theta_{\mu\nu}$.

To define unrenormalized quantities I use dimensional regularization[8] throughout.

Standard considerations say that introduction of a counterterm entails a corresponding renormalized parameter to compensate for the arbitrariness in the renormalization prescription. Since the operator $\varphi_0{}^2$ is multiplicatively renormalized, by a factor[9] $Z_m Z^{-1}$, I write

$$H_0 = (G + h_R Z_m)Z^{-1} . \tag{4}$$

Here h_R is the "renormalized improvement coefficient", and G is a counterterm, to be defined, which is independent of h_R. Gravity[10] couples to matter through $\theta_{\mu\nu}$, so we have a new coupling between gravity and matter. (The term in the Lagrangian is[1-4] $-\tfrac{1}{2}H_0 R\varphi_0{}^2$, where R is the scalar curvature.) Thus it is desirable to find some natural criterion to fix the improvement term. I will consider four criteria:

(i) That[11] in curved space-time the kinetic energy term in (1) be conformal invariant. In n space-time dimensions this gives $H_0 = \tfrac{1}{4}(n-2)/(n-1)$.

(ii) That[1] H_0 be such that when $n \le 4$ the operator $\theta_\mu{}^\mu$ is soft according to an uncritical application of the argument of Ref. 1. Again $H_0 = \tfrac{1}{4}(n-2)/(n-1)$. If $n=4$, the argument is fallacious.[5] This is manifest, for, in effect, the authors of Ref. 1 assume that $(n-4)g_B\varphi^4$ is zero at $n=4$, whereas in fact $g_B\varphi^4$ has a divergence there. However, the argument yields a unique value for H_0 at $n=4$. The question of whether $\theta_{\mu\nu}$ is thereby made finite has not been given a satisfactory answer in previous work.[1-6]

(iii) The finite improvement program[1-4]: Choose H_0 finite, such that $\theta_{\mu\nu}$ is finite in perturbation theory.

(iv) The renormalization-group (RG) covariant $\theta_{\mu\nu}$: Replace G by $G + Z_m k(g_R)$. Choose k as a finite function such that, when $h_R = 0$, a change in the renormalization mass μ needs no compensating change of h_R, but only of g_R, m_R, and the scale of φ (as usual).

I will prove that the last two criteria are equivalent, at $n=4$. Also, they agree with the first two if 't Hooft's methods[12] are used to sum the divergences. Further, the finite improvement program has a unique solution: H_0 depends only on n and equals $\tfrac{1}{4}(n-2)/(n-1)$ plus nonzero corrections of $O((n-4)^3)$. Thus criteria (i) and (ii) work up to the three-loop order, but not beyond.[13]

First, consider

$$G_{N\mu\nu}(q;p)\equiv\int d^4y\prod_{i=1}^{N}\int d^4x_i\exp(iy\cdot q+i\sum x_i\cdot p_i)\langle T\theta_{\mu\nu}(y)\prod\varphi(x_i)\rangle\,. \tag{5}$$

I define Z, Z_m, g_B, and G by 't Hooft's pole-part prescription,[12] with unit of mass μ. Also, to accommodate the RG-covariance definition of $\theta_{\mu\nu}$, I set

$$H_0=\{G+[k(g_R)+h_R]Z_m\}/Z\,. \tag{6}$$

Then the RG equation for $G_{N\mu\nu}$ is

$$\left(\mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial g_R}-\gamma_m m_R{}^2\frac{\partial}{\partial m_R{}^2}+\frac{1}{2}N\gamma-\delta\frac{\partial}{\partial h_R}\right)G_{N\mu\nu}=0\,. \tag{7}$$

Here β, γ_m, and γ are as usual,[12,14] and

$$\delta=\zeta(g_R)+\gamma_m h_R+\beta k'+\gamma_m k\,, \tag{8}$$

where

$$\zeta=\gamma_m Z_m{}^{-1}G+\beta(Z_m{}^{-1}G)'\,. \tag{9}$$

The prime denotes differentiation with respect to g_R. Note that G is independent[15] of m_R and μ, and ζ depends only on g_R. To eliminate the $\partial/\partial h_R$ term in Eq. (7) when $h_R=0$ and $n=4$, we must have

$$\bar\beta\partial k/\partial g_R+\gamma_m k=-\zeta\,, \tag{10}$$

where $\bar\beta$ is β evaluated at $n=4$. The boundary condition I choose is that k is a power series in g_R. At this point I only wish to consider defining the RG-covariant $\theta_{\mu\nu}$ for $n=4$; this means that $\bar\beta$ and not β appears in Eq. (10).

Earlier work[16] and a simple one-loop computation (for ζ) give the following lowest order values:

$$\bar\beta=\tfrac{3}{16}g_R{}^2/\pi^2+O(g_R{}^3),\quad \gamma_m=-\tfrac{1}{16}g_R/\pi^2+O(g_R{}^2),\quad \zeta=\tfrac{1}{96}g_R/\pi^2+O(g_R{}^2)\,. \tag{11}$$

Then

$$k=\left\{\frac{1}{6}-g_R{}^{1/3}\int_0^{g_R}\frac{dg}{g^{4/3}}\left[\frac{g\zeta(g)}{\bar\beta(g)}f(g)-\frac{1}{18}\right]\right\}[f(g_R)]^{-1}, \tag{12}$$

where

$$f(g_R)=\exp\int_0^{g_R}dg[\gamma_m(g)/\bar\beta(g)+\tfrac{1}{3}g^{-1}]\,. \tag{13}$$

Note that $k(0)=\frac{1}{6}$, which gives agreement in lowest order with Ref. 1.

To prove equality with the Callan, Coleman, and Jackiw (CCJ) definition,[1] consider the difference between the RG-covariant $\theta_{\mu\nu}$ and the value with a general H_0, viz.,

$$(GZ_m{}^{-1}+k-H_0ZZ_m{}^{-1})(\partial_\mu\partial_\nu-g_{\mu\nu}\Box)N[\varphi^2]\,. \tag{14}$$

Here the normal product $N[\varphi^2]$ (in n dimensions) is defined by Collins and by Breitenlohner and Maison,[17] and G is as before.

Now, by the definitions[12,14] of γ and γ_m,

$$(\beta\partial/\partial g_R+\gamma_m)Z_m{}^{-1}Z=0\,. \tag{15}$$

This and the definition (9) of ζ give

$$ZZ_m{}^{-1}=\exp[-\textstyle\int_0^{g_R}dg\,\gamma_m(g)/\beta(g)]\,, \tag{16}$$

$$GZ_m{}^{-1}=ZZ_m{}^{-1}\textstyle\int_0^{g_R}dg\,Z^{-1}Z_m\zeta(g)/\beta(g)\,. \tag{17}$$

So far, the equations above have been considered in their usual perturbation-theoretic sense as formal power series in g_R. However, it is attractive to assume that they make sense in the exact theory (if any). Then extracting the leading behavior of $ZZ_m{}^{-1}$ and $GZ_m{}^{-1}$ as $n\to 4$ gives

$$\tfrac{1}{6}ZZ_m{}^{-1}-GZ_m{}^{-1}=k+O((n-4)^{1/3})\,. \tag{18}$$

Then (14) shows that the CCJ[1] $\theta_{\mu\nu}$ is finite and equal to the RG-covariant $\theta_{\mu\nu}$, since[18] it has $H_0=\frac{1}{6}+O(n-4)$. But this need not be true order by order.

Next, $\theta_{\mu\nu}$, with H_0 arbitrary, is

$$\text{finite}-Z_m{}^{-1}(ZH_0-G)(\partial_\mu\partial_\nu-g_{\mu\nu}\Box)N[\varphi^2]\,. \tag{19}$$

So to satisfy the finite improvement program, we will try to find H_0 as a power series in g_R and $n-4$ such that $Z_m{}^{-1}(ZH_0-G)$ is finite in perturbation theory. The coefficients may depend on m_R and μ.

Now Eqs. (11) and (15) show that

$$Z_m^{-1}Z=\left(1+\frac{3g_R}{16\pi^2(n-4)}\right)^{1/3}+\text{nonleading}. \tag{20}$$

Hence, if X is any power series in g_R and $n-4$, and if $XZ_m^{-1}Z$ is finite, then $X=0$. Thus, (a) if H_0 satisfies the finite improvement program, it is unique; (b) since Z_m, Z, and G are independent of m_R and μ, so is H_0; and (c) $\beta\partial H_0/\partial g_R=0$ and hence $\partial H_0/\partial g_R=0$. [Here Eqs. (9) and (15) were used.]

Now let

$$H_0=\sum_{N=0}^{N}\eta_N(n-4)^N;$$

define the η_N by requiring the single pole terms in $Z_m^{-1}(ZH_0-G)$ to vanish. From Eqs. (9) and (15) it follows that

$$(\beta\partial/\partial g_R+\gamma_m)(Z_m^{-1}ZH_0-GZ_m^{-1})=-\zeta. \tag{21}$$

This equation shows that, if the single poles vanish, so do all the higher poles. Therefore the finite improvement program works. Since H_0 is a coefficient of the bare field, its success in being finite is independent of the renormalization prescription.

Since the value of $Z_m^{-1}(ZH_0-G)$ at $n=4$ satisfies the same equation and boundary condition as k, it must equal k. Hence criteria (iii) and (iv) agree, in perturbation theory.

Next I show that H_0 differs from $\frac{1}{4}(n-2)/(n-1)$ by terms of order $(n-4)^3$. The motivation is that a slight extension of Ref. 4 shows that taking $H_0=\frac{1}{4}(n-2)/(n-1)$ renormalizes $\theta_{\mu\nu}$ at the three-loop level.[13] Such a result looks nonaccidental. But we will see it is a consequence of the *topology* of the *two*-loop self-energy graph.

To extract the relevant information efficiently, we study $\theta_\mu{}^\mu$. First, however, consider the renormalization of the dimension-four operators. Use of the equations of motion and taking of traces and divergences shows that there is only one independent renormalization (besides Z_m, Z, and g_B). Take it to be A in

$$\tfrac{1}{24}\mu^{4-n}g_RN[\varphi^4]=\tfrac{1}{2}A\Box\varphi^2+\tfrac{1}{24}g_Rg_B'\varphi^4-\tfrac{1}{2}g_RZ'(\partial\varphi)^2+\tfrac{1}{2}m_R^2g_RZ_m'\varphi^2. \tag{22}$$

It has a corresponding RG coefficient

$$\alpha\equiv Z_m^{-1}[(A\beta/g_R)'-\gamma A/g_R+\tfrac{1}{2}\gamma'(Z_m-Z)]. \tag{23}$$

Define

$$D=(n-1)H_0-\tfrac{1}{4}n+\tfrac{1}{2}, \tag{24}$$

$$E=(A\beta/g_R-\tfrac{1}{2}\gamma Z)/Z_m. \tag{25}$$

Then

$$\beta E'+\gamma_mE=\beta(\alpha-\tfrac{1}{2}\gamma'), \tag{26}$$

$$\theta_\mu{}^\mu=\tfrac{1}{24}\mu^{4-n}(2g_R\gamma-\beta)N[\varphi^4]-\tfrac{1}{2}\gamma N[(\partial\varphi)^2]+\tfrac{1}{2}m_R^2(2+\gamma+\gamma_m)N[\varphi^2]+(DZ_m^{-1}Z+\tfrac{1}{2}E+\tfrac{1}{4}\gamma)\Box N[\varphi^2], \tag{27}$$

where the equation of motion has been used.

Let the first nonzero term in $\alpha-\frac{1}{2}\gamma'$ be of order g_R^m. Then Eq. (26) shows that the lowest-order divergent term in E is $O(g_R^{m+2})$ and is a single pole. Hence D is $O((n-4)^{m+1})$ to make $\theta_\mu{}^\mu$ finite. Topology of one- and two-loop self-energy graphs shows that $m\geq 2$. Explicit calculations up to three loops give

$$\alpha=\tfrac{1}{6}g_R/(16\pi^2)^2-\tfrac{17}{48}g_R^2/(16\pi^2)^3+O(g_R^3), \tag{28}$$

$$\gamma=\tfrac{1}{6}g_R^2/(16\pi^2)^2-\tfrac{1}{8}g_R^3/(16\pi^2)^3+O(g_R^4), \tag{29}$$

so that $m=2$.

Finiteness of $\theta_{\mu\nu}$ is equivalent[1-4] to finiteness of $\theta_\mu{}^\mu$. So the conformal value of H_0 makes $\theta_{\mu\nu}$ finite up to three loops and correct up to two, but not beyond.

Finally, I show that $\theta_{\mu\nu}$ defined above has a soft trace at a fixed point (i.e., if $\overline{\beta}=0$). This statement and its proof below depend on assuming that the equations used, which are derived in perturbation theory, are valid in the exact theory. Schroer[6] proved that $\theta_\mu{}^\mu$ can be soft only if $\overline{\beta}=0$, and for one value of h_R. So I put $H_0=(G+kZ_m)/Z$ in Eq. (27) to get

$$\theta_\mu{}^\mu=-\tfrac{1}{24}\mu^{4-n}(\beta+4g_RY)N[\varphi^4]+YN[(\partial\varphi)^2]+m_R^2(\tfrac{1}{2}\gamma_m+1-Y)N[\varphi^2]. \tag{30}$$

Here the equation of motion was used and

$$Y \equiv E + 2(n-1)(k + GZ_m^{-1}) + (1 - \tfrac{1}{2}n)Z_m^{-1}Z. \tag{31}$$

Use of Eqs. (9), (10), (15), and (26) gives

$$\gamma_m Y = -\bar{\beta}(2\zeta/g_R + \tfrac{1}{2}\gamma_m/g_R + 6k') + O(n-4). \tag{32}$$

So at a fixed point $Y=0$ if $\gamma_m \neq 0$. Hence $\theta_\mu{}^\mu$ is soft there, when $n=4$.

Fuller details of this work will appear elsewhere.

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[18]Note that $ZZ_m^{-1} \sim (n-4)^{-1/3}$, by Eq. (16).

Exact Equivalence of Chromodynamics to a String Theory*

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A previously proposed model of hadrons constructed out of quarks and strings is further developed into a fully interacting theory in any dimension. It is shown that in two dimensions chromodynamics is equivalent to this theory in the sense that the hadronic spectrum and matrix elements for the strong, weak, and electromagnetic interactions are *identical* in both theories.

According to some clues,[1] in an exact color SU(3) gauge theory (chromodynamics), the dynamics of confinement may be similar to the dynamics of the string model. These clues, combined with the intuition conveyed by duality diagrams, were the main motivations of previous work,[2] where the first principles for constructing a detailed model of hadrons out of quarks and strings were studied.

In this paper the model is further developed by adding a new term to the action which permits the hadrons to interact via a local interaction of the constituents. The resulting fully interacting model de-