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Forms of Gauge Fields and Nonintegrable Phase Factors*

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I show the connection between the magnetic monopoles with quantized strength and the form of solutions of gauge fields. A generalized electromagnetic field tensor $\bar{f}_{\mu\nu}$ and its physical implications in a unified gauge theory are discussed.

Recently, an excellent global formulation of gauge fields and an intrinsic description of electromagnetism were discussed by Yang and Wu on the basis of the concept of nonintegrable phase factors.[1,2] This concept, in its global ramification, leads to the interesting result that any gauge field must have a magnetic monopole of quantized strength, which depends only on the type of global gauge. The global gauge is a natural generalization of the usual concept to deal with the intricacies of the monopole field. It is remarkable that the mathematics of the global formulation of general gauge fields are wonderfully related to the topology of the fiber bundle, a natural geometrical concept.

In this paper, these results are derived and understood within the usual local gauge formulation in terms of the forms of gauge fields. I also discuss a generalized electromagnetic field tensor $\bar{f}_{\mu\nu}$ which satisfies the Maxwell equations everywhere except at the position of the monopole for the static case. The electric charge and the magnetic charge play symmetrical roles,[3] in contrast to the discussion in Ref. 1. Furthermore, the quantization condition for the monopole strength leads to the prediction $\sin^2\theta = \frac{1}{2}$ for the mixing angle in Weinberg's unified theory.[4] This can be tested experimentally. These discussions shed light on the connection between the global and the local properties of gauge fields.

Let us consider the SU(2) gauge theory involving the Yang-Mills field $b_\mu{}^k(x)$.[2] The phase factor is defined by[1]

$$\exp[ie\oint b_\mu{}^k(\tau_k/2)\,dx^\mu], \tag{1}$$

where τ_k are the Pauli matrices. I show that if $b_\mu{}^k$ is changed in the following way,

$$b_\mu{}^k \to b_\mu{}^k + b'_\mu{}^k, \tag{2}$$

$$b'_\mu{}^k(x) = -e^{-1}\epsilon^{kij}v^i(x)\partial_\mu v^j(x), \tag{3}$$

where $i,j,k = 1,2,3$, $v^i(x)$ is single valued, and $v^i v^i = 1$, then the phase factor (1) is unchanged. Using a generalization of Stokes's theorem, one has

$$\exp(\tfrac{1}{2}ie\oint b'_\mu{}^k\tau_k dx^\mu) = \exp(\tfrac{1}{2}ie\int dS^{\nu\mu}\partial_\nu b'_\mu{}^k\tau_k) = \lim_{R\to\infty}\exp[-\tfrac{1}{2}i\int_{S_R{}^2}(d\sigma)_\mu\epsilon^{0\mu\alpha\beta}\vec{\tau}\cdot(\partial_\alpha\vec{v}\times\partial_\beta\vec{v})]. \tag{4}$$

The two-dimensional sphere $S_R{}^2$ with radius $x^i x^i = R^2$ can be expressed in terms of two parameters q_a: $x^i = x^i(q_a)$, where $a = 1,2$. Since

$$(d\sigma)_k = \tfrac{1}{2}\epsilon_{kij}(\partial x^i/\partial q_a)(\partial x^j/\partial q_b)\epsilon_{ab}dq_1\,dq_2, \quad (\tfrac{1}{2}\tau\circ\epsilon_{ab}\partial_a\vec{v}\times\partial_b\vec{v})^2 = \det(\partial_a\vec{v}\cdot\partial_b\vec{v}),$$

Eq. (4) can be written as

$$\exp(\tfrac{1}{2}ie\oint b_\mu{}^k\tau_k dx^\mu)=\exp[-i\int d^2q(\det|\partial_a\vec{v}\cdot\partial_b\vec{v}|)^{1/2}\vec{\tau}\cdot\vec{\eta}]=\exp(-i4\pi n\vec{\tau}\cdot\vec{\eta})=1, \tag{5}$$

where the integer n is a wrapping number and $\vec{\eta}$ is a unit vector perpendicular to $S_R{}^2$ and independent of q_a.

Note that the constant factor in the solutions of the form (3) is completely determined by the nonlinear Yang-Mills field equations.[5] For example, the static Wu-Yang solution takes the form[5]

$$b'_0{}^k=0,\quad b'_j{}^k=-\epsilon_{jki}x^i/er^2,\quad r^2=x^ix^i. \tag{6}$$

This spherically symmetric solution is the same as (3) with $v^i(x)=x^i/r$. Of course, (3) is not the only form for the Yang-Mills field. We may also have the form

$$b_0{}^k=x^kG(r)/r,\quad b_j{}^k=\epsilon_{jki}x^iB(r)/r, \tag{7}$$

for the static spherically symmetric solutions. A particular complex solution is[6]

$$\begin{aligned}B(r)&=(\beta r-\sinh\beta r)/er\sinh\beta r,\\ G(r)&=i(\beta r\cosh\beta r-\sinh\beta r)/er\sinh\beta r,\end{aligned} \tag{8}$$

where β is a complex number with $\mathrm{Re}\beta\neq 0$.

To understand the physical meaning of the classical solution of the form (3), I define a generalized electromagnetic field tensor $\bar{f}_{\mu\nu}$ as the SU(2) field strength along the direction $v^k(x)$:

$$\bar{f}_{\mu\nu}=(\partial_\mu b_\nu{}^k-\partial_\nu b_\mu{}^k+e\epsilon^{kij}b_\mu{}^ib_\nu{}^j)v^k, \tag{9}$$

where $b_\mu{}^3$ is regarded as the usual electromagnetic potential $A_\mu(x)$.[3,1] Note that the definition (9) is different from that defined by 't Hooft.[3] The electromagnetic field tensor (9) is a natural generalization of the usual $F_{\mu\nu}$ used to deal with the intricacies of the monopole field. I believe that this is a necessary concept to understand the physics (i.e., electromagnetism) of gauge theories. This is to be contrasted with the usual non-Abelian analog of the electric and the magnetic fields.[5]

When the solutions of the Yang-Mills field $b_\mu{}^k$ take the form (3), we have

$$\bar{f}_{\mu\nu}=-e^{-1}\epsilon^{kij}v^k\partial_\mu v^i\partial_\nu v^j. \tag{10}$$

I can define the electric current j_μ and the magnetic current k_ν by

$$\bar{j}_\mu=\partial^\nu\bar{f}_{\mu\nu},\quad k_\mu=\tfrac{1}{2}\epsilon^{\mu\nu\alpha\beta}\partial_\nu\bar{f}_{\alpha\beta}, \tag{11}$$

which are obviously conserved, $\partial^\mu j_\mu=\partial^\mu k_\mu=0$. The total magnetic charge e_m related to $\bar{f}_{\mu\nu}$ is

$$e_m=\int dS^{\mu\nu}\bar{f}_{\mu\nu}/4\pi=-n/e, \tag{12}$$

where n is an integer related to the topological structure of $v^k(x)$. To see this, write the unit vector v^k in the form $v^k(x)=u^k(x)/[u^i(x)u^i(x)]^{1/2}$. Suppose that $x=x'$ is an isolated zero of $u^k(x)$, i.e., $u^k(x')=0$. The integer n is the Poincaré-Hopf index of the zero x'. The Wu-Yang solution (6) corresponds to $v^i(x)=x^i/r$, and hence $u^i=x^i$ has an isolated zero at $x^i=0$ which has the Poincaré-Hopf index $n=1$. Thus, the SU(2) gauge field defined on the global gauge $\mathcal{G}_n$ discussed by Wu and Yang[1] is the field with the form (3) in which the Poincaré-Hopf index of the isolated zero x' of $u^k(x)$ is n. Also, the condition (5) for $b_\mu{}^k$ in SU(2) is equivalent to the requirement that in overlapping regions $(b_\mu{}^k)_a$ and $(b_\mu{}^k)_b$ be related by a single-valued global gauge transformation as discussed in Ref. 1. I stress that for the solution (6), the electromagnetic field tensor $\bar{f}_{\mu\nu}$ satisfies the Maxwell equations, i.e., Eq. (11) with $j_\mu=k_\mu=0$, everywhere except at the position of the monopole $x^k=0$.

I show that the quantization condition in gauge theories can give interesting predictions. Let us consider a nontrivial SU(2) ⊗ U(1) gauge theory, i.e., the Weinberg unified theory with bosons only, for simplicity.[4] The Lagrangian involves the photon field A_μ, the neutral massive Z_μ, and the charged fields W_μ with mass M_W,

$$\begin{aligned}W^{+\mu}&=(b_1{}^\mu\mp ib_2{}^\mu)/\sqrt{2},\\ Z^\mu&=b_3{}^\mu\cos\theta-B^\mu\sin\theta,\\ A^\mu&=b_3{}^\mu\sin\theta+B^\mu\cos\theta,\end{aligned} \tag{13}$$

and some scalar fields φ_1^0, φ_2^0, and $\varphi^\pm$, where $b_k{}^\mu$ and B^μ are the SU(2) and the U(1) gauge fields, respectively. Because the electromagnetic field A^μ is given by the combination $b_3{}^\mu\sin\theta+B^\mu\cos\theta$, it is natural to define the generalized electromagnetic field tensor $\bar{F}_{\mu\nu}$ in such SU(2) ⊗ U(1) theory as

$$\bar{F}_{\mu\nu}=(\partial_\mu b_\nu{}^k-\partial_\nu b_\mu{}^k+g\epsilon^{kij}b_\mu{}^ib_\nu{}^j)v^k\sin\theta+(\partial_\mu B_\nu-\partial_\nu B_\mu)\cos\theta \tag{14}$$

in analogy with (9), where g is related to the charge e by $e=-g\sin\theta$ in the theory. A particular static spherically symmetric solution is given by[6]

$$\varphi^{\pm} = \varphi_2^0 = 0, \quad \varphi_1^0 = -2\sqrt{2}M_W/g,$$
$$B^0 = 0, \quad B^i = (x^i/r)\bar{B}(r), \tag{15}$$
$$b_0^k = 0, \quad b_i^k = -\epsilon_{ikj}x^j/gr^2,$$

where $\bar{B}(r)$ is arbitrary. This can be readily verified. It follows from (14) and (15) that the magnetic charge $\bar{e}_m$ is

$$\bar{e}_m = \int \bar{F}_{\mu\nu}\, dS^{\mu\nu}/4\pi = -(\sin\theta)/g = (\sin^2\theta)/e, \tag{16}$$

which is not quantized, in contrast to the case discussed above. If we assume the Dirac quantization condition,[7] i.e., $\bar{e}_m e = \frac{1}{2}$, we obtain

$$\sin^2\theta = \tfrac{1}{2}, \tag{17}$$

for the mixing angle θ in Weinberg's unified theory. The motivation of this assumption is as follows: In U(1) gauge field theory, such as electromagnetism, one has naturally the Dirac quantization condition $e_m e = n/2$ (n an integer) for the magnetic charge e_m, which corresponds to classification of the U(1) bundle according to the first Chern class; yet in SU(2) gauge field theory, one has the condition $e_m e = n$ (n an integer).[1] These can also be domonstrated by explicit solutions of gauge fields. However, the solution (15) for SU(2) $\otimes$ U(1) gauge fields does not automatically satisfy these two conditions as shown in (16). Since the results of Wu and Yang indicate that any gauge field must have a magnetic monopole of quantized strength,[1] it is natural and necessary to explore the physical consequences of the magnetic charge $\bar{e}_m$ satisfying these quantization conditions. It turns out that if $\bar{e}_m$ in Eq. (16) satisfies the condition $\bar{e}_m e = 1$, one gets the result $\cos\theta = 0$ which destroys the desired unification in Weinberg's theory. Thus we can only assume Dirac's condition $\bar{e}_m e = \frac{1}{2}\bar{e}_m$. The result (17) is interesting and should be tested experimentally.[8]

It has been stressed that electromagnetism is the gauge invariant manifestation of the nonintegrable phase factor $\exp(ie \oint A_\mu\, dx^\mu)$, which provides an intrinsic and complete description of electromagnetism.[1] To illustrate a basic difference between the U(1) and the SU(2) gauge theories, let us consider the static field $A_\mu(x)$ with the form, in analogy with (3),

$$A_0 = 0, \quad A_i = (1/2e)\epsilon^{3ij}\epsilon_{ab}W_a\partial_j W_b, \tag{18}$$

where $W_1(x) = (r - x_3)^n = 1/W_2(x)$. It is readily checked that

$$\exp(ie \oint A_\mu\, dx^\mu) = \exp(-2\pi n i) = 1. \tag{19}$$

Furthermore, the field (18) satisfies the Maxwell equations everywhere except at $x_k = 0$ and it is related to the magnetic monopole with the strength $e_m = n/e$ sitting at the origin. In sharp contrast with the SU(2) case, the constant factor $1/2e$ in (18) cannot be determined by the dynamical equations for the sourceless U(1) gauge field. In general, one may have $e_m = n/2e$ or other values by changing the constant factor in (18) or the power n in $W_a(x)$. A similar situation also occurs in the global formulation for the U(1) gauge field.[1,9] This suggests that the quantization of the monopole strength in SU(2) gauge theory is more intimately related to dynamics than that in U(1) gauge theory. Finally I remark that in the global formulation the quantization condition is closely related to the global gauge transformation. Yet from the viewpoint of local gauge formulation discussed above, the quantization condition and the dynamical equations which determine $v^i(x)$ in (3) are interlocked.

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[7]P. A. M. Dirac, Proc. Roy. Soc. London, Ser. A 133, 60 (1931). Note that (16) in Weinberg's theory is incompatible with the Schwinger condition $\bar{e}_m e = 1$. The result (16) has also been obtained in Ref. 6 which has *nothing to do with the concept of the nonintegrable phase factors.*

[8]Because of the great difficulty in measuring θ, one

should not allow oneself to be too discouraged simply because (17) did not agree with some experiments.

[9]It may be argued that if Dirac's quantization condition $e_m e = n/2$ is not satisfied then the field of a magnetic monopole of strength e_m cannot be taken as a realizable physical situation. See Ref. 1.

Finite Improvement Renormalizes the Energy-Momentum Tensor*

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The problem of the improvement term of the energy-momentum tensor $\theta_{\mu\nu}$ in φ^4 theory is reconsidered. A unique finite improvement coefficient is shown to renormalize $\theta_{\mu\nu}$. Dimensional regularization is used and the improvement coefficient depends only on the space-time dimension. Up to three-loop order but not beyond, the value suggested by conformal arguments works. But if use is allowed of 't Hooft's methods to sum the divergences, then this value does work.

The problem of renormalizing the energy-momentum tensor in φ^4 theory has often been considered in the past.[1-6] As I will show, improvements to the treatment can be made.

In terms of the bare field φ_0, the Lagrangian is

$$\mathcal{L} = \tfrac{1}{2}(\partial\varphi_0)^2 - \tfrac{1}{2}m_0^{\,2}\varphi_0^{\,2} - \tfrac{1}{24}g_0\varphi_0^{\,4}. \tag{1}$$

Then the canonical energy-momentum tensor is[7]

$$T_{\mu\nu} = \partial_\mu\varphi_0\partial_\nu\varphi_0 - g_{\mu\nu}\mathcal{L}. \tag{2}$$

To renormalize the connected Green's functions of $T_{\mu\nu}$, it is sufficient[1-4] to define an improved energy-momentum tensor

$$\theta_{\mu\nu} \equiv T_{\mu\nu} - H_0(\partial_\mu\partial_\nu - g_{\mu\nu}\Box)\varphi_0^{\,2}, \tag{3}$$

and to choose the improvement coefficient H_0 appropriately. I ignore the renormalization needed of the vacuum expectation value of $\theta_{\mu\nu}$.

To define unrenormalized quantities I use dimensional regularization[8] throughout.

Standard considerations say that introduction of a counterterm entails a corresponding renormalized parameter to compensate for the arbitrariness in the renormalization prescription. Since the operator $\varphi_0^{\,2}$ is multiplicatively renormalized, by a factor[9] $Z_m Z^{-1}$, I write

$$H_0 = (G + h_R Z_m)Z^{-1}. \tag{4}$$

Here h_R is the "renormalized improvement coefficient", and G is a counterterm, to be defined, which is independent of h_R. Gravity[10] couples to matter through $\theta_{\mu\nu}$, so we have a new coupling between gravity and matter. (The term in the Lagrangian is[1-4] $-\tfrac{1}{2}H_0 R\varphi_0^{\,2}$, where R is the scalar curvature.) Thus it is desirable to find some natural criterion to fix the improvement term. I will consider four criteria:

(i) That[11] in curved space-time the kinetic energy term in (1) be conformal invariant. In n space-time dimensions this gives $H_0 = \tfrac{1}{4}(n-2)/(n-1)$.

(ii) That[1] H_0 be such that when $n \leq 4$ the operator $\theta_\mu{}^\mu$ is soft according to an uncritical application of the argument of Ref. 1. Again $H_0 = \tfrac{1}{4}(n-2)/(n-1)$. If $n=4$, the argument is fallacious.[5] This is manifest, for, in effect, the authors of Ref. 1 assume that $(n-4)g_B\varphi^4$ is zero at $n=4$, whereas in fact $g_B\varphi^4$ has a divergence there. However, the argument yields a unique value for H_0 at $n=4$. The question of whether $\theta_{\mu\nu}$ is thereby made finite has not been given a satisfactory answer in previous work.[1-6]

(iii) The finite improvement program[1-4]: Choose H_0 finite, such that $\theta_{\mu\nu}$ is finite in perturbation theory.

(iv) The renormalization-group (RG) covariant $\theta_{\mu\nu}$: Replace G by $G + Z_m k(g_R)$. Choose k as a finite function such that, when $h_R = 0$, a change in the renormalization mass μ needs no compensating change of h_R, but only of g_R, m_R, and the scale of φ (as usual).

I will prove that the last two criteria are equivalent, at $n=4$. Also, they agree with the first two if 't Hooft's methods[12] are used to sum the divergences. Further, the finite improvement program has a unique solution: H_0 depends only on n and equals $\tfrac{1}{4}(n-2)/(n-1)$ plus nonzero corrections of $O((n-4)^3)$. Thus criteria (i) and (ii) work up to the three-loop order, but not beyond.[13]