PHYSICAL REVIEW LETTERS

VOLUME 36 | 14 JUNE 1976 | NUMBER 24

Classical Particlelike Behavior of Sine-Gordon Solitons in Scattering Potentials and Applied Fields*

M. B. Fogel, S. E. Trullinger,† A. R. Bishop, and J. A. Krumhansl

Laboratory of Atomic and Solid State Physics, and Materials Science Center, Cornell University, Ithaca, New York 14853

(Received 29 March 1976)

We show that classical sine-Gordon solitons maintain their integrity to a high degree in the presence of external perturbations. Two examples, of particular importance in condensed matter, are described in detail: (i) A model impurity is found to bind low-velocity solitons but merely phase shift those with high velocities, and (ii) external static driving terms with damping accelerate the soliton to a terminal velocity. The importance of a translation mode is emphasized and it is concluded that the soliton behaves as a classical particle in all essential respects.

Nonlinear fields,[1] particularly those exhibiting solitary-wave solutions, are now of interest in many physical problems. Free-field solutions have been investigated extensively, particularly for quantum fields[2]; but in condensed matter we require knowledge of the behavior and integrity of solitons in the presence of "impurities" or applied fields. Kink solutions, or domain walls, occur in magnetic[3] and ferrodistortive[4] materials and in many Landau-Ginzburg expansion contexts.[5]

In this Letter we describe results of a general perturbation method for examining these kinds of problems. The technique is applicable to nonlinear equations possessing translational invariance. For definiteness we consider the case of a sine-Gordon (SG) soliton and examine the effect on its motion of (i) a weak model impurity potential, and (ii) a constant external driving term and a viscous term in the SG equation of motion.

For (i) we find that high-velocity solitons pass through the impurity region with only a phase shift while those with low velocity become trapped and oscillate. In case (ii) we find transient perturbations which decay rapidly for a large damping constant and leave a soliton moving with a terminal velocity determined by the relative magnitude of damping and forcing constants. *In all cases we conclude that the soliton retains a localized shape and that its dynamics are essentially those of a classical Newtonian particle.*

The pure SG equation[1] is a nonlinear wave equation of Lorentz-covariant form:

$$\frac{\partial^2\psi}{\partial t^2} - c_0^2\frac{\partial^2\psi}{\partial x^2} + \omega_0^2\sin\psi = 0. \tag{1}$$

Here c_0 and ω_0 are a velocity and frequency characteristic of the particular physical context.[1-5] The various solutions of (1) are known exactly.[1-6] For instance

$$\psi_\pm^v(x,t) = 4\tan^{-1}\exp[\pm(\omega_0/c_0)\gamma(x-vt)], \tag{2}$$

with $\gamma \equiv (1-v^2/c_0^2)^{-1/2}$, are soliton (+) and antisoliton (−) solutions traveling with velocity v ($|v| < c_0$). Linearized perturbations $\varphi(x,t)$ about $\psi_\pm^v$ have been studied extensively, e.g., to establish linear stability.[1,7] They may be written as $\varphi(x,t) = f(x)e^{-i\omega t}$, where $f(x)$ satisfies a one-dimensional Schrödinger-like eigenvalue equation:

$$-c_0^2\frac{d^2f}{dx^2} + \omega_0^2\left(1 - 2\,\mathrm{sech}^2\frac{\omega_0}{c_0}x\right)f = \omega^2 f. \tag{3}$$

The assumption $v=0$ made here is completely general since a Lorentz transformation to the soliton rest frame is always possible. Equation (3) has exactly[7] one "bound" eigenstate with $\omega_b^2=0$ and

$$f_b(x)=\frac{2\omega_0}{c_0}\operatorname{sech}\frac{\omega_0}{c_0}x. \qquad (4)$$

The remaining eigenfunctions form a continuum with $\omega_k^2=c_0^2k^2+\omega_0^2$ and

$$f_k(x)=(2\pi)^{-1/2}\frac{c_0}{\omega_k}e^{ikx}\left(k+i\frac{\omega_0}{c_0}\tanh\frac{\omega_0}{c_0}x\right). \qquad (5)$$

These continuum solutions resemble those of the linearized version of (1) (Klein-Gordon equation), except for a perturbation localized around the soliton and an asymptotic phase shift.[7,8] The zero-frequency bound state (4) is of paramount importance to our discussion and may be viewed as a Goldstone mode because the soliton breaks the continuous translational symmetry. Since $f_b(x)=\partial\psi_+^0/\partial x$, $\psi(x)=\psi_+^0(x)+af_b(x)$ corresponds (in the linear order considered) to a soliton translated by a distance $-a$. Thus $f_b(x)$ may be termed the "translation mode," and allows us to describe the soliton motion.[9] The continuum solutions correspond to small perturbations of the soliton shape.

The essence of our technique is to recognize that $f_i(x)$ $(i=b,k)$, being eigenfunctions of a self-adjoint operator, form a complete set. Their orthogonality and completeness relations are readily established.[7] We repeatedly used this complete set to expand linearized perturbations about various functions.[10]

(i) *Impurity-soliton interaction.*—Here we consider the Hamiltonian density

$$\mathcal{H}(x)=A\left[\frac{1}{2}\left(\frac{\partial\psi}{\partial t}\right)^2+\tfrac{1}{2}c_0^2\left(\frac{\partial\psi}{\partial x}\right)^2+\omega_0^2(1-\cos\psi)-\lambda\frac{\partial\psi}{\partial x}g(x)\right], \qquad (6)$$

i.e., the usual SG density[1] plus a term representing the interaction[11] of ψ with the impurity potential $g(x)$. The constant A sets the energy scale, and for simplicity we take $g(x)=\theta(x-x_0)-\theta(x+x_0)$. This simple step function form, chosen for analytic tractability, is not a limitation of our method. It is expected to exhibit all of the qualitative features of more realistic impurity potentials. In the following the coupling constant λ is assumed small.

We first form the equation of motion for ψ from (6):

$$\frac{\partial^2\psi}{\partial\tau^2}-\frac{\partial^2\psi}{\partial z^2}+\sin\psi=\alpha\left[\delta(z+z_0)-\delta(z-z_0)\right], \qquad (7)$$

where $\tau\equiv\omega_0 t$, $z\equiv\omega_0 x/c_0$, $\alpha\equiv\lambda/(c_0\omega_0)$. A soliton (or antisoliton) given by (2) and initially moving with velocity v will be modified if $\alpha\neq 0$. Labeling this perturbation by $\varphi(z,\tau)$, we assume $\varphi\ll 1$ and using (7) form the equation governing it to linear order. Upon Lorentz transforming to the rest frame of the unperturbed soliton this becomes (with $\beta\equiv v/c_0$)

$$\frac{\partial^2\varphi}{\partial\tau^2}-\frac{\partial^2\varphi}{\partial z^2}+(1-2\operatorname{sech}^2 z)\varphi=\frac{\alpha}{\gamma}\left[\delta\left(z+\beta\tau+\frac{z_0}{\gamma}\right)-\delta\left(z+\beta\tau-\frac{z_0}{\gamma}\right)\right]. \qquad (8)$$

Equation (8) can be solved using the Fourier time transform

$$\varphi(z,\bar\omega)=(2\pi)^{-1/2}\int_{-\infty}^{+\infty}d\tau\, e^{i\bar\omega\tau}\varphi(z,\tau) \quad (\bar\omega\equiv\omega/\omega_0), \qquad (9)$$

and expanding $\varphi(z,\bar\omega)$ in the complete set (4) and (5):

$$\varphi(z,\bar\omega)=\varphi_b(\bar\omega)f_b(z)+\int_{-\infty}^{+\infty}d\kappa\,\varphi(\kappa,\bar\omega)f_\kappa(z) \quad (\kappa\equiv kc_0/\omega_0). \qquad (10)$$

After projection and inverse Fourier transformation we find $\varphi_b(\tau)$ and $\varphi(\kappa,\tau)$. Even for the step function form of $g(z)$ these are cumbersome expressions, and here we simply summarize their physical interpretation.

Since f_b is the translation mode, $\varphi_b(\tau)$ describes the "center-of-mass" motion of the soliton in the presence of the impurity. We find that as the impurity approaches $z=0$ (soliton center) in the rest frame the soliton begins to be affected, and when the leading edge passes $z=0$ the soliton acquires a velocity $\beta^*=\alpha/8\beta$ in the negative direction (for $\alpha>0$) which is retained until the trailing edge passes $z=0$. Thus for $\alpha>0$ (<0) the soliton slows down (speeds up) in the impurity region. (Precisely the opposite effects occur for the antisoliton.) The soliton acquires an asymptotic phase shift $\delta=\frac{1}{4}\alpha z_0\beta^{-2}$ in the lab frame. The continuum contributions in (10) consist of two pieces (both proportional to α): one localized in the impurity region and persisting for all time; the other occurring only while the soliton is near the impurity and describing a slight distortion of the soliton wave form. We have illustrated some

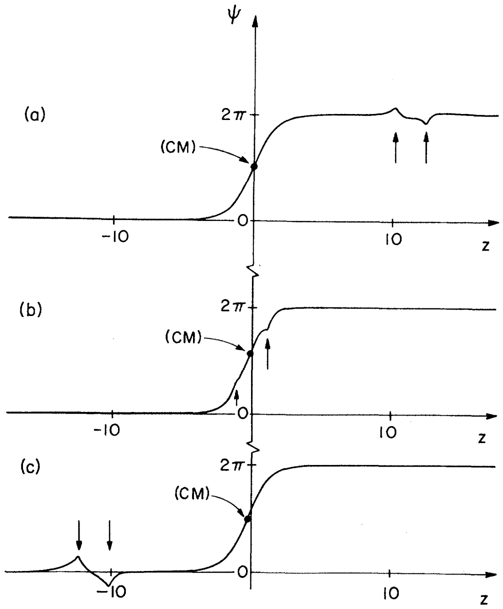

FIG. 1. A representative soliton-impurity collision is shown in the initial rest frame of the soliton (a) before, (b) during, and (c) after the interaction. The parameter values are $\alpha = 1.5$, $\beta = 0.9$, and $z_0 = 5$. The "center-of-mass" (CM) position of the soliton suffers a phase shift (see text). The vertical arrows indicate the boundaries of the "Lorentz-contracted" impurity region.

of this behavior in Fig. 1.

Our results show that φ diverges as β^{-2}, for $\beta \to 0$, so that the linear perturbation theory is invalid for low velocities even for $\alpha \ll 1$. This suggests that low-velocity solitons may become trapped or repelled by the impurity. We have examined this possibility by considering a soliton at rest a distance $\xi c_0/\omega_0$ from the impurity center ($z=0$), and computing from (6) the change $\Delta V(\xi)$ in the static soliton energy due to the impurity:

$$\Delta V(\xi) = 4A\omega_0 c_0 \alpha \tan^{-1}(\sinh z_0/\cosh\xi). \qquad (11)$$

Note that $\Delta V(\xi=0)$ is a minimum (maximum) for $\alpha<0$ (>0), with opposite conditions for the antisoliton. We have also found the classical condition for binding ($\alpha<0$) or reflecting ($\alpha>0$) a soliton by comparing the initial soliton kinetic energy with $|\Delta V(0)|$. Trapping or reflection occurs for $\beta^2 < \beta_c^2 \equiv 1 - [1+\frac{1}{2}|\alpha|\tan^{-1}(\sinh z_0)]^{-2}$.

If the soliton behaved as a Newtonian particle (of mass[7] $M = 8A\omega_0 c_0$), it would execute oscillatory motion about $\xi = 0$ (for $\alpha<0$, $\beta^2<\beta_c^2$) described by $M\partial^2\xi/\partial\tau^2 = -\partial(\Delta V)/\partial\xi$. We studied this in two regimes where the soliton width is (a) small and (b) large compared with the impurity width.

In case (a) we sought harmonic oscillations (at least for small β) determined by expanding $\Delta V(\xi)$ to quadratic order in ξ, i.e., $\xi(\tau) = \xi_0 \sin(\Omega\tau)$ with $\Omega^2 = \frac{1}{2}|\alpha| \tanh z_0/\cosh z_0$. In case (b) we anticipated that the finite soliton extent would result in distinctly anharmonic oscillations. Indeed expanding $\Delta V(\xi)$ to quartic order and assuming Newtonian motion led to a nonlinear oscillator equation with Jacobi elliptic functions as oscillatory solutions,[4] which have the expected character.

We studied the validity of the Newtonian *Ansatz* by considering $\psi = \psi_+^0(z-\xi(\tau)) + \varphi(z,\tau)$. The first term describes a soliton oscillating about $z=0$ and φ is the difference between this *Ansatz* and the exact solution ψ to (7). Anticipating small φ we again linearized and expanded in the basis functions (4) and (5). For both the limits (a) and (b) above the amplitudes $\varphi_b(\tau)$, $\varphi(\kappa,\tau)$ were found to be small, as required for consistency.

(ii) *External fields and damping.*—Consider the equation of motion

$$\frac{\partial^2\psi}{\partial t^2} - c_0^2\frac{\partial^2\psi}{\partial x^2} + \omega_0^2 \sin\psi = E - \eta\frac{\partial\psi}{\partial t}, \qquad (12)$$

where $\eta\,\partial\psi/\partial t$ is a viscous term and E is a constant driving force. For small E, we proceed in the same spirit as for (i), assuming a solution to (12) of the form $\psi(z,\tau) = \psi_\pm^v(z-\beta\tau) + \varphi(z,\tau)$, linearizing in φ and transforming to a coordinate frame moving with velocity v. The perturbation φ satisfies

$$\frac{\partial^2\varphi}{\partial\tau^2} - \frac{\partial^2\varphi}{\partial z^2} + (1-2\,\mathrm{sech}^2 z)\,\varphi + \gamma\,\Gamma\frac{\partial\varphi}{\partial\tau} - \beta\gamma\,\Gamma\frac{\partial\varphi}{\partial z} = \chi + 2\beta\gamma\,\Gamma\,\mathrm{sech}z, \qquad (13)$$

with $\Gamma \equiv \eta/\omega_0$, $\chi \equiv E/\omega_0^2$. We again expand $\varphi(z,\tau)$ in the complete set (4) and (5) and project out the amplitudes $\varphi_b(\tau)$ and $\varphi(\kappa,\tau)$ [cf. (10)].

The solutions will be presented in detail elsewhere. Here we note only that $\varphi_b(\tau)$ satisfies[12]

$$\partial^2\varphi_b/\partial\tau^2 + \gamma\,\Gamma\,\partial\varphi_b/\partial\tau = \beta\gamma\Gamma + \tfrac{1}{4}\pi\chi. \qquad (14)$$

Assuming $[\partial\varphi_b/\partial\tau]_{\tau=0} = 0$, i.e., a soliton initially at rest in the frame moving with velocity v, and perturbations turned on at $\tau=0$, we find from (14) that $\varphi_b(\tau) = (\beta + \frac{1}{4}\pi\chi\gamma^{-1}\Gamma^{-1})[\tau - \gamma^{-1}\Gamma^{-1}(1-e^{-\gamma\Gamma\tau})]$. Since $\varphi_b(\tau)$ is the translation mode amplitude we see that the soliton achieves a terminal velocity ($-z$ direction) $\beta^T = \beta + \frac{1}{4}\pi\chi\gamma^{-1}\Gamma^{-1}$. If $\chi=0$ then $\beta^T = \beta$ and the soliton comes to rest in the lab frame.[13] Thus the soliton behaves precisely as a Newtonian particle moving in a force field and viscous medium. Indeed (14) is just Newton's second law for the position $(-\varphi_b)$ of such a particle. Except for some initial transient time dependence, the continuum contributions to $\varphi(z,\tau)$, describing modifications of the soliton shape, are localized about the soliton[14] and are time independent.

In conclusion we have illustrated a simple perturbative method for examing the influence of external perturbations on soliton[15] or solitary-wave solutions of nonlinear wave equations. In translationally invariant cases we find that the solitons behave essentially as classical[16] extended particles. This conclusion includes the "φ^4" problem,[2] although there the analog of (3) has an extra bound state to which external perturbations can couple. These results may be applied, for example, to Bloch walls in impure materials, magnetic solitons[17] in ^{3}He, and "charged-phase solitons"[5] in one-dimensional charge-density-wave condensates.

Note Added:—We have recently become aware of numerical studies of the damping problem carried out by Nakajima and co-workers.[18] Our results agree with theirs for small values of the driving force.

*Work supported in part by the U. S. Energy Research and Development Administration under Contract No. AT(11-1)-3161, Technical Report Number COO-3161-39, and also through the Cornell Materials Science Center Report No. 2604.

†National Science Foundation Postdoctoral Fellow.

[1]For a review, see A. C. Scott, F. Y. F. Chu, and D. W. McLaughlin, Proc. IEEE 61, 1443 (1973). In some cases the solitary waves are "solitons."

[2]R. Rajaraman, Phys. Rep. 21C, 229 (1975).

[3]A. R. Bishop, E. Domany, and J. A. Krumhansl, to be published.

[4]J. A. Krumhansl and J. R. Schrieffer, Phys. Rev. B 11, 3535 (1975).

[5]M. J. Rice, A. R. Bishop, J. A. Krumhansl, and S. E. Trullinger, Phys. Rev. Lett. 36, 432 (1976).

[6]D. W. McLaughlin, J. Math. Phys. 16, 96 (1975).

[7]Julio Rubinstein, J. Math. Phys. 11, 258 (1970).

[8]Note that their dispersion relation is unaffected by the soliton; the "potential" in Eq. (3) is "reflectionless" (see Ref. 7). The bound state corresponds to a binding of the $k=0$ mode by the soliton.

[9]The translation mode has also required special attention in field-theory quantization procedures (see Ref. 2).

[10]The inverse integral transform of a function $F(k)$ has to be replaced by its Cauchy principal value if $F(k)$ has a singularity on the real k axis.

[11]Other interaction forms can be treated; this one was motivated by Ref. 5.

[12]In fact there are coupling terms between φ_b and $\varphi(\kappa)$ which result in a "polarization" of the soliton by a cloud of linearized solutions. These have been omitted from Eq. (13) since they are of order $\beta\varphi(\kappa)$.

[13]Even though $\varphi_b(\tau)$ becomes large for $\tau \gg 1$, absorbing $\varphi_b(\tau)$ into the argument of $\psi_\pm(z)$ is equivalent to the procedure of removing secular terms and makes the perturbation theory valid for large τ. See A. H. Nayfeh, *Perturbation Methods* (Wiley, New York, 1973), Chap. 3.

[14]Except for a small uniform shift of the "equilibrium" value of ψ.

[15]A. Newell (private communication) has pointed out that it is possible to use the inverse scattering method (see Ref. 1) to treat certain types of perturbations.

[16]It seems likely that a Schrödinger equation involving a wave function of φ_b would constitute a quantum generalization; this is an interpretation of recent attempts to quantize the translation mode in field theory (see Ref. 2).

[17]Kazumi Maki and Hiromichi Ebisawa, to be published.

[18]K. Nakajima, Y. Sawada, and Y. Onodera, J. Appl. Phys. 46, 5272 (1975), and Refs. 18 and 19 therein.