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Ising-Model Critical Indices in Three Dimensions from the Callan-Symanzik Equation*

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The coefficients in the Callan-Symanzik equations for a three-dimensional, continuous spin Ising model with an $\exp(-As^4+Bs^2)$ spin-weight factor are expanded in the dimensionless, renormalized coupling constant. These series are summed by the Padé-Borel method to yield the critical indices $\gamma=1.241\pm0.002$, $\eta=0.02\pm0.02$, $\nu=0.63\pm0.01$, and $\Delta_1=0.49\pm0.01$.

The renormalization-group approach to critical phenomena introduced by Wilson[1] and substantially elaborated by others[2-6] has contributed a great deal of physical understanding. It has contributed calculations, via the ϵ-expansion approach of Wilson and Fisher[7] and of Brézin and co-workers,[6,8,9] of fair accuracy. Somewhat improved accuracy is obtained by Colot, Loodts, and Brout.[10] So far, however, the accuracy of the calculations is substantially inferior to that of the high-temperature-series approach,[11] and inadequate for detailed comparison with experiments. Here we will show that by the introduction of appropriate and sufficiently powerful series summation techniques, the approach advocated by Parisi[12,13] can yield results of accuracy comparable to high-temperature-series results. Specifically we will treat the three-dimensional, continuous spin Ising model via the renormalized, perturbation expansion of the Callan[14]-Symanzik[15] equation. We have been able to compute β to sixth order and the other coefficients to fifth order. This treatment involves the unproven assumption that the right-hand side of the Callan-Symanzik equation is asymptotically negligible. It is known that this result is true order-by-order in perturbation theory. Our results for this treatment of this model are

$$\begin{aligned} v^* &= 1.423\pm0.01\,,\\ \eta_4 &= 1/\nu-2+\eta = -0.3843\pm0.003\,,\\ \gamma &= 1.2410\pm0.002\,,\\ \omega &= 0.78\pm0.01\,, \end{aligned} \tag{1}$$

where v^* is the fixed-point value of the renormalized coupling constant; ν is the exponent for the correlation length, $\xi=(T-T_c)^{-\nu}$; η is the low-frequency, magnetic susceptibility index, $\chi\propto k^{\eta-2}$ at $T=T_c$; and γ is the static magnetic susceptibility index, $\chi\propto(T-T_c)^{-\gamma}$. Finally, ω is Wegner's[16] correction to scaling index. Since scaling holds explicitly for this theory one may compute from Eq. (1) by the usual relations

$$\begin{aligned} \eta &= 0.021\pm0.02\,,\\ \nu &= 0.627\pm0.01\,,\\ \Delta_1 &= \omega\nu = 0.49\pm0.01\,. \end{aligned} \tag{2}$$

The results quoted in (1) and (2) are all in agreement with the best high-temperature-series results[11,17] for the spin-$\frac{1}{2}$ Ising model except for

γ which is significantly lower than the accepted value of

$$\gamma = 1.250 \pm 0.003 . \tag{3}$$

We do not know whether this difference is intrinsic, the result of bias in the numerical estimates, or the result of the approximation made in deriving the Callan-Symanzik equation.

In order to study the model described by the partition function

$$Z = \int_{-\infty}^{+\infty} \cdots \int \left(\prod d\sigma_{\vec{i}}\right) \exp\Big[\sum_{\text{lattice}} \Big(\sum_{\vec{\delta}} K\sigma_{\vec{i}}\sigma_{\vec{i}+\vec{\delta}} - A\sigma_{i}^{4} + B\sigma_{i}^{2}\Big)\Big], \tag{4}$$

where $\vec{\delta}$ ranges over half the set of nearest-neighbor lattice sites, in terms of the above-mentioned techniques, we find it convenient to recast Eq. (4) in a more suggestive notation. Let us make the replacements

$$\vec{i} = \Lambda\vec{x}, \quad \Delta x = 1/\Lambda ; \quad \sigma_{\vec{i}} = \zeta s_{\vec{i}}, \quad (\partial s)^2 = \sum_{\vec{\delta}} (s_{\vec{i}} - s_{\vec{i}+\vec{\delta}})^2/(\Delta x)^2 . \tag{5}$$

In this notation Eq. (4) becomes

$$Z = \int_{-\infty}^{+\infty} \cdots \int \prod (ds_{\vec{i}}\zeta) \exp\Big\{-\tfrac{1}{2}\Big[\sum_{\text{lattice}} Z_3(\partial s)^2 + m^2 s^2\Big](\Delta x)^d - \sum_{\text{lattice}} \big[g(Z_1/4!)s^4 + \tfrac{1}{2}\delta m^2 s^2\big](\Delta x)^d\Big\} , \tag{6}$$

where

$$Z_3 = K\zeta^2\Lambda^{d-2}, \quad Z_1 = (4!)A\zeta^4\Lambda^d/g ,$$
$$m^2 + \delta m^2 = -2\Lambda^2(d + B/K)Z_3 , \tag{7}$$

with d the space dimension of the hypercubical lattice on which the model is specified. We have, by this substitution, introduced three arbitrary parameters which we may select to suit our purposes. The form (6) is a lattice version of Euclidean boson, $\lambda\varphi^4$ field theory. Renormalization theory for this situation assures us that if we define the renormalization constants Z_1, Z_3, and m^2 by the conditions

$$\Gamma^{(2)}(p, -p; m, g)\big|_{p^2=0} = m^2 ,$$
$$\left.\frac{\partial\Gamma^{(2)}(p, -p; m, g)}{\partial p^2}\right|_{p^2=0} = 1 , \tag{8}$$
$$\Gamma^{(4)}(0, 0, 0, 0; m, g) = g ,$$

we obtain a finite limit as the scale factor $\Lambda \to \infty$. The functions $\Gamma^{(N)}$ are the vertex functions, or one-particle irreducible Green's functions. In the resulting theory the spin-spin correlation decays at large distances like

$$\exp(-m|\vec{x}|) = \exp(-|\vec{i}|m/\Lambda) . \tag{9}$$

Since we are concerned with the critical point, we want the correlation length, which Eq. (9) implies is Λ/m, to tend to infinity.

Following Brézin, Le Guillou, and Zinn-Justin,[6] we introduce the dimensionless "coupling constant," u, by

$$g = m^{4-d}u .$$

Then they[6] showed that the Callan-Symanzik equation

$$\left(m\frac{\partial}{\partial m} + \beta(u)\frac{\partial}{\partial u} - \frac{1}{2}N\eta(u)\right)\Gamma^{(N)} = \Delta\Gamma^{(N)} \simeq 0 \tag{10}$$

is satisfied. The right-hand side is asymptotically, as $\Lambda/m \to \infty$, smaller by a factor $(m/\Lambda)^2$, up to powers of $\ln(\Lambda/m)$, order by order in perturbation theory and is assumed not to affect the critical behavior in what follows. Hubbard[18] has discussed this assumption.

Analysis of the Callan-Symanzik equation leads to the following prescription for computing the critical indices of the model (4). First $\beta(u)$ is expected to behave as sketched in Fig. 1. The critical-point behavior is determined by the behavior of the coefficients at the zero u^* of $\beta(u)$. The coefficients in (10) are defined by

$$\beta(u) = (d-4)\left(\frac{\partial \ln[uZ_1(u)/Z_3^2(u)]}{\partial u}\right)^{-1} , \tag{11}$$

$$\eta(u) = \beta(u)\,\frac{\partial \ln Z_3(u)}{\partial u} . \tag{12}$$

In addition it is convenient to define an additional

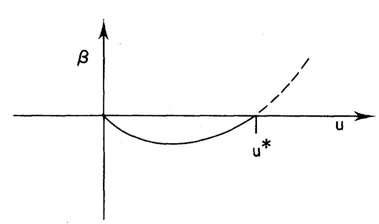

FIG. 1 Expected behavior of $\beta(u)$.

renormalization constant associated with the $s_{\bar{x}}^2$ vertex

$$1 = Z_4(u)\Gamma^{(1,2)}(p; q, -q; m, g)|_{p=q=0}, \tag{13}$$

where $\Gamma^{(M,N)}$ has M distinct s^2 vertices and N distinct s vertices. Associated with $Z_4(u)$ is the additional function

$$\eta_4(u) = \beta(u)\,\frac{\partial \ln Z_4}{\partial u}. \tag{14}$$

The critical indices are given by

$$\begin{aligned} &\beta(u^*) = 0, \\ &1/\nu - 2 + \eta = \eta_4(u^*), \quad \eta = \eta(u^*), \\ &\gamma^{-1} = 1 + \eta_4(u^*)/[2 - \eta(u^*)], \\ &\omega^* = \beta'(u^*). \end{aligned} \tag{15}$$

The expansion of β, η, and η_4 in powers of u is computable in terms of the renormalized perturbation theory. Parisi[13] and Brézin, Le Guillou, and Zinn-Justin[8] have computed these series for general dimension and number of components thru orders three, two, and two, respectively (Parisi's work is unfortunately marred by misprints). We have extended these results by three additional orders for $d = 3$ and the single-component system described in (4). The extension to multicomponent systems is not hard.

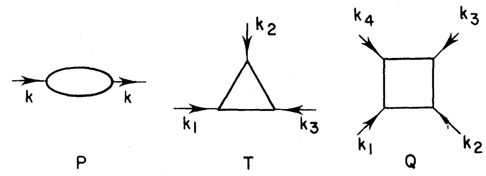

FIG. 2. Single-loop insertions with two, three, and four external momenta.

A technical point which greatly facilitates these higher-order calculations is that the necessary single-loop graphs can be evaluated analytically. For the graphs pictured in Fig. 2 we obtain

$$P = \pi^{-2}\int d^3K\{(1+\vec{K}^2)[1+(\vec{K}+\vec{k})^2]\}^{-1} = (2/k)\tan^{-1}(k/2), \tag{16}$$

$$T = \pi^{-2}\int d^3K\{(1+\vec{K}^2)[1+(\vec{K}+k_1)^2][1+(\vec{K}+k_1+k_2)^2]\}^{-1} = \Delta^{-1}\tan^{-1}(\Delta/C), \tag{17}$$

where

$$\Delta^2 = \begin{vmatrix} 1 & 1+\frac{1}{2}k_1^2 & 1+\frac{1}{2}k_3^2 \\ 1+\frac{1}{2}k_1^2 & 1 & 1+\frac{1}{2}k_2^2 \\ 1+\frac{1}{2}k_3^2 & 1+\frac{1}{2}k_2^2 & 1 \end{vmatrix}, \quad C = 4 + \tfrac{1}{2}k_1^2 + \tfrac{1}{2}k_2^2 + \tfrac{1}{2}k_3^2, \tag{18}$$

and

$$Q = \frac{1}{\pi^2}\int d^3K\{(1+\vec{K}^2)[1+(\vec{K}+\vec{k}_1)^2][1+(\vec{K}+\vec{k}_1+\vec{k}_2)^2][1+(K+k_1+k_2+k_3)^2]\}^{-1} = \sum_{i=1}^{4}\frac{F_i}{2D\Delta_i}\tan^{-1}\left(\frac{\Delta_i}{C_i}\right), \tag{19}$$

where the F_i and D are 4×4 determinants, Δ_i^2 is a 3×3 determinant, and C_i is a polynomial as in (18). The full details of the evaluations will be left to a separate paper.[19]

We remark, by way of motivation, that by the general theory of graphs with four lines joining at each vertex we expect of the order of $(2n)!$ graphs in nth order, but that the contribution (except for certain less numerous subsets of graphs) of each graph will be of order $1/(n!)$. Thus we expect the series for β, η, and η_4 to be divergent like $n!$ In order to sum this type of series efficiently the Padé-Borel method[20] is most appropriate. This combines the Borel summation procedure,

$$\sum_{n=0}^{\infty} a_n x^n = \int_0^{\infty} e^{-t}\left[\sum_{n=0}^{\infty}\frac{a_n(tx)^n}{n!}\right]dt, \tag{20}$$

with the well-known Padé approximant mehod[20] for the analytic continuation of the function in square brackets in Eq. (20) to the range $0 \le t < \infty$. The procedure is applicable when the poles of the Padé approximant do not lie in the right half-plane, and Eq. (20) may be evaluated in terms of exponential integrals thru a partial fraction expansion of the Padé approximant.

If we let $u = 16\pi v/3$, and $\beta(v) = 3\beta(u)/16\pi$ for a convenient numerical scale, then we obtain

$$\beta(v) = -v + v^2 - 0.422\,496\,572 v^3 + 0.351\,069\,598 v^4 - 0.376\,526\,8177 v^5 + 0.4960 v^6 + \dots, \tag{21}$$

$$\eta_4(v) = -\tfrac{1}{3}v + \tfrac{2}{27}v^2 - 0.044\,310\,2537 v^3 + 0.039\,519\,5663 v^4 - 0.044\,384 v^5 + \dots, \tag{22}$$

$$[\gamma(v)]^{-1} = 1 - \tfrac{1}{6}v + \tfrac{1}{27}v^2 - 0.023\,069\,621 v^3 + 0.019\,886\,819 v^4 - 0.022\,451 v^5 + \dots. \tag{23}$$

The Padé-Borel analysis of β is given in Table I. On the basis of Table I, and the examination of the convergence of table f values of the function $\beta(v)$ we obtain the estimates of Eq. (1). These estimates are confirmed by direct Padé approximation when the Padé approximants have at least two nontrivial zeros, but with an uncertainty perhaps 5 times larger. For a discussion of this type of error analysis, see Hunter and Baker.[21]

TABLE I. Analysis of $\beta(v^*)$.

Approximant	v^*	$\beta'(v^*)$
[2/1]	1.5967	0.6536
[3/1]	1.4161	0.7826
[4/1]	1.4299	0.7660
[5/1]	1.4152	0.7908
[3/2]	1.4287	0.7680
[4/2]	1.4229	0.7760

For the analysis of η_4, four Padé-Borel approximants are available, [2/1], [3/2], [4/1], and [3/2]. By considering the apparent convergence of the table of values and the uncertainty caused by a possible error in v^* we obtain the estimate of Eq. (1). The estimate is again confirmed by direct Padé analysis which gives $\eta_4 \sim 0.386 \pm 0.005$.

The analysis of series for η,

$$\eta(v) = 0.010\,973\,936v^2 + 0.000\,914\,221v^3 + 0.001\,796\,224v^4 - 0.000\,4169v^5 + \dots, \tag{24}$$

is not, at this order, susceptible to an improved rate of convergence by means of the Padé-Borel method as it does not alternate in sign. Consequently, we have found it more advantageous to analyze $1/\gamma(v)$. In this case the [2/1], [3/1], [4/1], [2/2], and [3/2] approximants are available. A review of the apparent convergence and the uncertainty in v^* leads to the estimate of Eq. (1). Direct Padé-approximant estimates are consistent in all cases with the results noted in (1) and (2), but with greater uncertainty.

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