Critical Exponents for the Three-Dimensional Ising Model from the Real-Space Renormalization Group in Two Dimensions

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The equivalence of the critical behavior of the d-dimensional Ising model at a finite temperature to that of a $d-1$ dimensional system at zero temperature as a function of a transverse magnetic field is used to calculate the critical exponents for the *three-dimensional* model by applying the Niemeijer–Van Leeuwen renormalization-group method to the *two-dimensional* problem.

The renormalization-group method of Niemeijer and Van Leeuwen[1-3] has proved very powerful in yielding critical exponents as well as transition temperatures for various two-dimensional lattices. A direct application of the method into three dimensions is, however, quite clumsy. A method is presented here by which critical indices for the *three-dimensional* Ising model are obtained, applying the Niemeijer–Van Leeuwen technique to a *two-dimensional* Ising system in a transverse magnetic field, at zero temperature.

The equivalence of the critical behavior of the d-dimensional Ising system as a function of temperature to that of a $(d-1)$-dimensional system, at zero temperature, as a function of a transverse magnetic field, has been rigorously proved by Suzuki[4] for $d=2$. An exact solution for the resulting one-dimensional problem was first given by Pfeuty,[5] and more recently by Stoeckly and Scalapino.[6] The generalization to $d>2$ has been conjectured by Pfeuty and Elliott,[7] and is strongly supported by their numerical results. It can also be justified following the arguments in the Stoeckly and Scalapino[6] paper and in a paper by Dietrich.[8] One starts from a field theory in d dimensions, which is then reduced to $d-1$ dimensions, using a continuum generalization of the transfer-matrix technique.[9] The equivalence of the resultant $(d-1)$-dimensional problem to an Ising model in a transverse field can then be shown for the original parameters at a certain limit, similar to that taken when the discrete Ising Hamiltonian is to be recovered from a continuous one.[10] The critical behavior of the system is, however, unaffected in this limit if universality is assumed. In the equivalent $(d-1)$-dimensional problem, the transverse field, rather than the temperature, is now the disordering agent; the temperature now corresponds to the inverse length along the dth dimension of the original system if periodic boundary conditions are assumed. An infinite d-dimensional Ising system is thus equivalent to a $d-1$ dimensional system at *zero temperature* in a transverse field. I mention here that a most thorough discussion of the transfer-matrix technique and its continuous generalization can be found in a series of articles by Camp and Fisher.[11-15] They point out the equivalence of the d-dimensional classical problem to a $(d-1)$-dimensional quantum field theory[11,14] for general d and mention also the relation to a system in a perpendicular field.

In this note, I treat the $d=3$ case by generalizing the technique of Niemeijer and Van Leeuwen[1-3] to the quantum-mechanical problem of an Ising Hamiltonian containing a transverse magnetic field term that does not commute with the rest of the Hamiltonian.

The Hamiltonian is

$$\mathcal{H} = -\Gamma\sum_{i=1}^{N}\sigma_i^x - \frac{J}{2}\sum_{\langle i\neq j\rangle}\sigma_i^z\sigma_j^z, \tag{1}$$

where σ_i^x and σ_i^z are spin-$\frac{1}{2}$ Pauli matrices, the first sum is over all N sites of a two-dimensional lattice, and the second sum is on all nearest-neighbor pairs i, j. The order parameter $\langle\sigma^z\rangle$ will be zero for transverse fields larger than a certain critical value Γ_c. Note, however, that $\langle\sigma^x\rangle\neq 0$ for a transverse field in the x direction.

As in Ref. 1, a triangular lattice is chosen. The latter is then divided into triangular cells having three sites. The resulting lattice of cells is again triangular as illustrated in Fig. 1. At zero temperature, the thermodynamic properties are obtainable from the diagonal matrix elements of the projection operator $P(\mathcal{H}) = |0\rangle\langle 0|$, where

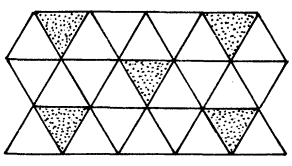

FIG. 1. Triangular lattice with cells shaded.

$|0\rangle$ is the ground state of $\mathcal{H}$. Note that $P(\mathcal{H})$ depends on the ratio J/Γ only. A basis state is an external product $|\sigma_1^z\rangle|\sigma_2^z\rangle\ldots|\sigma_N^z\rangle$ of N eigenstates of the site operators σ_i^z. This state can also be written as a product of $(N/3)$ cell states, each of which is a product of three eigenstates of the operators σ_i^z belonging to the cell. There are eight basis states per cell, each of which is an eigenstate of the z component of the total spin of the cell, having some eigenvalue m. These states are then divided into two groups of four states, according to the sign of m. A typical state is denoted $|\sigma_\alpha^z, q_\alpha\rangle$, $\sigma_\alpha^z = \pm 1$, $q_\alpha = 0,1,2,3$, $\alpha = 1,\ldots,N/3$, σ_α^z now serving as a cell spin. I next define a renormalization-group transformation from the site-spin Hamiltonian $\mathcal{H}$ to the cell-spin Hamiltonian $\mathcal{H}'$ by the equation

$$\langle\sigma_1^z|\ldots\langle\sigma_\alpha^z|\ldots P'(\mathcal{H}')\ldots|\sigma_\alpha^z\rangle\ldots|\sigma_1^z\rangle = \sum_{\{q_\alpha\}}\langle\sigma_1^z,q_1|\ldots\langle\sigma_\alpha^z,q_\alpha)\ldots P(\mathcal{H})\ldots|\sigma_\alpha^z,q_\alpha\rangle\ldots|\sigma_1^z,q_1\rangle. \quad (2)$$

An exact solution of Eq. (2), for $\mathcal{H}'$, is generally more difficult than the exact diagonalization of the Hamiltonian in Eq. (1). An approximate solution can, however, be found using various approximation schemes, the simplest of which is probably one that uses a perturbation expansion for $P(\mathcal{H})$. We separate $\mathcal{H}$ into a piece $\mathcal{H}_0$ consisting of the single-spin terms plus all intracell interactions, and a perturbation V containing all the intercell interactions. A first-order perturbation expansion for $P(\mathcal{H})$ yields

$$P(\mathcal{H}) \cong |0\rangle\langle 0| + \sum_{n\neq 0}|0\rangle\langle 0|V\frac{|n\rangle\langle n|}{E_0 - E_n} + \sum_{n\neq 0}\frac{|n\rangle\langle n|}{E_0 - E_n}V|0\rangle\langle 0|, \quad (3)$$

where $|n\rangle$ is an eigenstate of $\mathcal{H}_0$ with an eigenvalue E_n, and 0 refers to the ground state of $\mathcal{H}_0$. It is important to note that the ground state $|0\rangle$ is nondegenerate and is invariant with respect to reversal of any number of cell spins σ_α^z. Substituting the expansion (3) into the right-hand side of Eq. (2) results in an equation for the diagonal matrix elements of $P'(\mathcal{H}')$. Using a similar perturbation expansion for $\mathcal{H}'$, where now $\mathcal{H}_0'$ contains all the single-cell-spin terms and V' all the interactions, one finds that if $\mathcal{H}$ contains only nearest-neighbor interactions, a possible solution, $\mathcal{H}'$, to Eq. (2) is

$$\mathcal{H}' = -\Gamma'\sum_{\alpha=1}^{N/3}\sigma_\alpha^x - \frac{J'}{2}\sum_{\langle\alpha,\beta\rangle}\sigma_\alpha^z\sigma_\beta^z, \quad (4)$$

where the second sum is on nearest-neighbor cells only. The zero-order terms in the perturbation expansions turn out to be identical (independent of J/Γ or J'/Γ'). By equating the first-order terms one gets the equation

$$K' = K'(K) = 32K\sum_{n,m\neq 0}\frac{\langle\varphi_0|\sigma^z|\varphi_n\rangle\langle\varphi_0|\sigma^z|\varphi_m\rangle}{(e_n - e_0)(e_m - e_0)}\Delta_{n0}\Delta_{m0}, \quad (5)$$

where $K = J/\Gamma$ and $K' = J'/\Gamma'$. $|\varphi_n\rangle$ is an eigenstate of a single-cell Hamiltonian with a corresponding energy e_n, and σ^z is a site-spin operator. Δ_{n0} is defined as

$$\Delta_{n0} = |\textstyle\sum_q\langle\varphi_n|\sigma, q\rangle\langle\sigma, q|\varphi_0\rangle|, \quad (6)$$

and 0 refers to the ground state of the cell Hamiltonian.

The solution to the equation $K^* = K'(K^*)$ defines the critical Hamiltonian K^*, while the "temperaturelike" eigenvalue is given by

$$\lambda_\Gamma = \partial K'(K)/\partial K|_{K=K^*}. \quad (7)$$

Equations (5) and (7) were solved numerically, and yielded $K^* = 0.3013$, and $\lambda_\Gamma = 1.7651$. This corresponds to a scaling index[16] $a_\epsilon = \ln\lambda_\Gamma/\ln 3 = 0.5172$ [$\epsilon = (\Gamma - \Gamma_c)/\Gamma_c$], which is to be compared to the value $a_\epsilon = 0.5333$, obtained from exact series calculations.[16]

In the second order, next-nearest-neighbor and next-next-nearest-neighbor interactions are also generated by the transformation. The fixed-point Hamiltonian was found to be $K^* = 0.2917$, $L^* = -0.0104$, and $M^* = -0.0092$, where L^* and M^* refer to next- and next-next-nearest-neighbor interactions. The temperaturelike eigenvalue was found to be $\lambda_\Gamma = 1.8715$, which corresponds to $a_\epsilon = 0.5368$.

The "magnetic-field-like" eigenvalue can be obtained from the response to a perturbation of the form $h\sum_i\sigma_i^z$ of the fixed-point Hamiltonian $\mathcal{H}(K^*)$. While the lowest-order calculation yielded $a_H = \ln\lambda_H/\ln 3 = 0.8424$ to be compared to $a_H = 0.8333$

from exact series results,[16] the deviation of a_H calculated to second order from the series result[16] is much larger ($a_H \simeq 0.91$). The reason for the divergence is unclear, but may be because of the fact that the number of intercell versus intracell interactions (two to one in this calculation of a_H) is rather large. It is thus hoped that suitable finite-cluster calculations, or choosing a larger cell size, will yield more satisfactory results. (The question of the convergence of the cumulant expansion and its dependence on the size of the cells has also been recently considered by Sudbø and Hemmer.[17])

Finally, it is of interest to see how the $d \to d-1$ crossover is described in this scheme. To this end, consider the $(d-1)$-dimensional system at a small but finite temperature $1/\beta$. This corresponds to a d-dimensional system of length $L_d \propto \beta$ along its dth dimension. The density operator is then

$$\rho \approx \frac{|0\rangle\langle 0| + 1\rangle\langle 1| e^{-\beta \Delta E} + \ldots}{1 + e^{-\beta \Delta E} + \ldots}, \tag{8}$$

where ΔE is the energy difference between the ground state $|0\rangle$ and the first excited state $|1\rangle$. As pointed out by Pfeuty and Elliott,[7] these two states become degenerate for $\Gamma = \Gamma_c$ and

$$\Delta E \propto (\Gamma - \Gamma_c)^{\nu_d}, \tag{9}$$

where ν_d is the d-dimensional critical exponent for the correlation length. We therefore have for the arguments of the exponentials in Eq. (8)

$$\beta \Delta E \propto \beta/(\Gamma - \Gamma_c)^{-\nu_d} \propto L_d/\xi_d, \tag{10}$$

where ξ_d is the d-dimensional correlation length; namely, the higher excited states of the $(d-1)$-dimensional system become important when the correlation length of the d-dimensional system becomes comparable with its length along the dth dimension. This is in accordance with the Fisher finite-size scaling arguments.[18]

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