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[21]The result presented in Eq. (4) is valid for a stream pulse that travels with uniform velocity through a uniform plasma; i.e., a pulse which is a function of z and t only in the combination $Vt-z$. It does not depend on the assumption of an infinite plasma, although the detailed forms of the various terms are influenced by the presence or absence of boundaries.
[22]These expressions were derived by neglecting the $\vec{j}_p \times \vec{B}$ term in Eq. (3). This is consistent with the results obtained: The stream is quickly current-neutralized except within a thin shell at its radius. When the injected current is large enough to make the net current in this shell significant, as is now being considered, the magnetic force causes the plasma electrons in this shell to antipinch. Consequently, I_{pr} increases and the critical current for magnetic self-focusing is lowered below the value given here. However, the concomitant increase in positive space charge in the shell limits this process and keeps the critical current from being lowered greatly.
[23]This result was derived, under the assumption that $a\omega_p >> 1$, by comparing the expression for $\tilde{\varphi}$ found in Ref. 6 with the expression for $\tilde{\rho}$ obtained by use of Eq. (10) in Ref. 7. The tilde denotes the Fourier transform with respect to η as defined in Ref. 6.

Strong-Turbulence Theory and the Transition from Landau to Collisional Damping

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New methods are introduced for a quantitative evaluation of the dielectric constant describing the interaction of a long-wavelength test wave with electrons in the presence of electron-ion collisions or small-scale turbulence. It is shown that the usual resonance-broadening arguments of strong-turbulence theory do not apply.

Collisional effects on wave propagation have been investigated by many authors. In the strongly collisional regime, $\omega/\nu \ll 1$, $k\lambda \ll 1$, the dispersion relation may be obtained from the two-fluid transport equations.[1] In the weakly collisional regime corrections to Landau damping have been found either from the Landau collision term by iteration or by use of model collision terms such as the Bernstein-Greene-Kruskal or Fokker-Planck term with constant diffusion and friction coefficients. In the case of ion-sound and related modes where electrons of all speeds $v > v_{\rm ph} \ll v_e$ can resonate with the wave, such procedures become dubious on two grounds.[2] The collision frequency for the dominant process, pitch-angle scattering by electron-ion collisions, is strongly velocity dependent, $\nu(v) = \nu_e (v_e/v)^3$, and iterative procedures cannot be applied to resonant particles.

The breakdown of iterative procedures for resonant particles is the starting point of Dupree's perturbation theory for strong turbulence[3] and related theories. A principal result of Dupree's theory is the *broadening* of wave-particle resonances $\omega - \vec{k}\cdot\vec{v} = 0$. The broadening is estimated as $\Delta\omega = (\frac{1}{3}k^2 D)^{1/3}$, where $D(v)$ is the velocity diffusion coefficient. Not even the solution of a simplified diffusion equation for the ensemble-averaged orbits has been obtained however. By various methods or simply by ignoring the velocity dependence of $\overleftrightarrow{D}(v)$ one arrives at

$$\exp[i\vec{k}\cdot\vec{x}(-t)] = \exp[i\vec{k}\cdot(\vec{x}-\vec{v}t) - \tfrac{1}{3}\vec{k}\cdot\overleftrightarrow{D}\cdot\vec{k}t] \quad (1)$$

replacing the usual unpertubed orbits. Accordingly, the usual resonant denominators are replaced by the Laplace transform of (1). Customarily even further approximations are made to replace $\delta(\omega - \vec{k}\cdot\vec{v})$ by a Lorentzian or square function of width $\Delta\omega$. While such considerations are perhaps sufficient to illustrate the origin of resonance broadening[3] they have recently been reiterated so many times that it becomes increasingly difficult to question their practical use. The principal aim of this paper is not only to do just that but to introduce new methods for obtaining quantitative results. Rather than trying to reduce the results of formal perturbation theory to a tractable problem we start from simple physical concepts, making use of the analogy between collisional and turbulent scattering of particles. That this is of more than heuristic value has been demonstrated in two earlier papers. It is frequently necessary to consider at the same time both collisions and turbulence effects[4] and it has been shown that the modified turbulence theory, mode coupling included, may be derived in exactly the same way as collisional modifications.[5] One perturbs the equation for the average distribution function, linearizing in the test-wave amplitude but *not* with respect to the background fluctuation spectrum. The test-particle propagator (conditional probability density) $P(\vec{x}, \vec{v}, t|\vec{x}', \vec{v}', t')$, $t' > t$, is the Green's function for the solution of the resulting equations. The modified quasilinear dielectric constant, e.g., takes the form

$$\epsilon^1(\vec{k}, \omega) = 1 - \sum_j i\left(\frac{\omega_j}{k}\right)^2 \int d^3v N(\vec{v}|\vec{k}, \omega)\vec{k}\cdot\frac{\delta f}{\delta\vec{v}}, \qquad (2)$$

where $N(\vec{v}|\vec{k}, \omega)$ is the Fourier-Laplace transform of the conditional probability density for the test-particle position $\vec{x}'(t')$,

$$N(\vec{v}|\vec{k}, \omega) = \int_0^\infty d\tau \int d^3r \int d^3v'\, P(\vec{x}, \vec{v}, t|\vec{x}+\vec{r}, t+\tau, v')\exp[i(\omega\tau - \vec{k}\cdot\vec{r})]. \qquad (3)$$

Note that we do not make the conventional approximation which neglects the action of the propagator on $\delta f/\delta\vec{v}$. This has been accomplished by the use of the adjoint propagator[4] which in the present case (homogenous plasma) amounts simply to an interchange of the $\vec{v}$ and $\vec{v}'$ integrations. Use of the adjoint propagator becomes even more important in the electromagnetic conductivity tensor where $\vec{k}\cdot(\delta f/\delta\vec{v})$ is replaced by a much more complicated expression. In this case one computes $\vec{V}(\vec{v}|\vec{k}, \omega) = \int d^3v'\, \vec{v}'\, P(\vec{v}|\vec{k}, \omega, \vec{v}')$. Generally, the method applies if one does not need to compute $f_1(\vec{k}, \omega, \vec{v})$ itself but only certain moments.

$N(\vec{v}|\vec{k}, \omega)$ has to be found from the kinetic equation with the appropriate turbulent and particle-particle collision terms. The procedure (approximation) depends on the specific problem to be considered. There seem to be no short cuts, such as suggested by (1). Generally, it is not useful to attempt an approximate solution for the complete test-particle propagator P but as much as possible one should apply approximation methods to the required moments of P. This will be illustrated by the specific case to be considered now, which I think is one of the simplest physically interesting and *consistent* problems. We study the interaction of electrons with a test wave in the presence of an isotropic low-phase-velocity, $v_{\rm ph} \ll v_e$ turbulent spectrum, for definiteness, e.g., both ion-sound test wave and spectrum.[6] The dominant effect of such fluctuations is pitch-angle scattering just as that of electron-ion collisions $[\nu_e = \pi\omega_e(W/nT_e)\langle\omega_e/kv_e\rangle]$. The equation for N takes then the form

$$-i(\omega - \vec{k}\cdot\vec{v})N(\vec{v}|\vec{k}, \omega) = \frac{\nu(v)}{2}\frac{\delta}{\delta\vec{v}}\cdot\left(\frac{\vec{\vec{I}} - \vec{v}\vec{v}}{v^2}\right)\cdot\frac{\delta N}{\delta\vec{v}} + 1, \qquad (4)$$

which is obtained by $\vec{v}'$ integration and Fourier-Laplace transform of the backward Kolmogorov equation for P describing the pitch-angle diffusion process. The test wave is restricted to low frequency and long wavelength for two reasons. Firstly, collisional effects are strongest in this case as may be seen by writing (4) in dimensionless form, $N = \hat{N}(\hat{\nu}, \hat{\omega}, \mu)/kv$, $\hat{\nu} = \nu/kv$, $\hat{\omega} = \omega/kv$, and $\mu = \vec{k}\cdot\vec{v}/kv$. Secondly, in general, the collision term would depend on frequency ω and wave number k of the test wave. Equation (4) is valid for $k \ll k'$ and $\omega - \vec{k}\cdot\vec{v} \ll k'v$, where ω' and k' are typical frequency and wave number of the background spectrum. The inelastic scattering processes lead to the establishment of a Maxwell distribution in the classical case to a *self-similar* distribution[7]

$$f(\vec{v}) = c_5\exp[-(v/v_0)^5] \qquad (5)$$

in the turbulent case. The effect of inelastic collisions on N may be neglected (take $Z \to \infty$ in the classical case).

Equations (2) and (4) are written in terms of spherical coordinates v, θ, and φ, $\cos\theta=\mu$, with k as polar axis. $\hat{N}(\hat{\nu},\hat{\omega},\mu)$ is expanded in Legendre polynomials $P_l(\mu)$, since they are eigenfunctions of the collision operator:

$$\hat{N}(\hat{\nu},\hat{\omega},\mu)=\sum_{l=0}^{\infty}(-i)^l(2l+1)\left(\frac{(\frac{1}{2}l-\frac{1}{2})!}{(\frac{1}{2}l)!}\right)\tfrac{1}{2}!\,N_l(\hat{\nu},\hat{\omega})P_l(\mu). \tag{6}$$

From (4) it follows that $f_l=N_{l+1}/N_l$ satisfies

$$f_l-(1/f_{l-1})+g_l=0,\quad l=1,2\ldots, \tag{7}$$

where $g_l=[-i\hat{\omega}+\frac{1}{2}\hat{\nu}l(l+1)]\frac{1}{2}(2l+1)[(\frac{1}{2}l-\frac{1}{2})!/(\frac{1}{2}l)!]^2$, $l=0,1,2\ldots$. For an isotropic distribution (2) becomes

$$\epsilon_e(\vec{k},\omega)=-(\omega_e/k)^2\int_0^\infty dv\,4\pi v(\delta f/\delta v)[1+i\hat{\omega}\hat{N}(\hat{\nu},\hat{\omega})], \tag{8}$$

where $\hat{N}(\hat{\nu},\hat{\omega})=\frac{1}{2}\int_{-1}^{1}d\mu\hat{N}(\hat{\nu},\hat{\omega},\mu)=\frac{1}{2}\pi N_0(\hat{\nu},\hat{\omega})$. N_0 can be written as a continued fraction, $N_0=(1/g_0+\ldots)\times(1/g_1+\ldots)\cdots$, which was evaluated numerically by a simple algorithm. The first two iterations yield in the strongly collisional regime $\hat{\nu}\gg1,\hat{\omega}$

$$N_0=\frac{2}{\pi}\frac{(\frac{1}{3}\hat{\nu})+i\hat{\omega}}{\hat{\omega}^2+(\frac{1}{3}\hat{\nu})^2}. \tag{9}$$

In the collisionless limit $\hat{\nu}\to0^+$

$$N_0{}^0=\frac{1}{\pi}\int_{-1}^{1}d\mu\,\frac{i}{\hat{\omega}-\mu+i0^+}=H(\hat{\omega})+\frac{i}{\pi}\ln\left|\frac{1+\hat{\omega}}{1-\hat{\omega}}\right|, \tag{10}$$

where $H(\hat{\omega})=1$ for $\hat{\omega}<1$, and 0 for $\hat{\omega}>1$.

$\hat{\omega}=1$ is the boundary between resonating and nonresonating particles. In the weakly collisional regime $\hat{\nu}\ll1$ the differential equation (4) may be solved by the methods of *singular* perturbation theory. For resonating particles $\hat{\omega}<1$ we introduce the stretched variable $\eta=(\hat{\omega}-\mu)/\epsilon$, $\epsilon=\hat{\nu}^{1/3}$, and find

$$\hat{N}(\hat{\nu},\hat{\omega},\eta)=(1/\epsilon)\int_0^\infty d\tau\exp[i\eta\tau-(1-\hat{\omega}^2)\tau^3/3][1+\epsilon F_1(\hat{\omega},\tau)\cdots], \tag{11}$$

where $F_1(\hat{\omega},\tau)$ are polynomials of degree $5l$ in τ. Equation (11) should be compared to (1). The resonance function (3) can be expressed in terms of Airy-Hardy integrals Ei_3 or Lommel functions. All we require however is the angle-averaged resonance function $\hat{N}(\hat{\nu},\hat{\omega})$ which can be obtained much more directly. It can be shown that $\epsilon\int_{-\infty}^{+\infty}d\eta\,\hat{N}=\pi$; thus $\hat{N}(\hat{\nu},\hat{\omega})=\pi/2-(\epsilon/2)(\int_{-\infty}^{(\hat{\omega}-1)/\epsilon}+\int_{(\hat{\omega}+1)/\epsilon}^{\infty}d\eta\,\hat{N})$, where outside the resonance region $\eta=O(1)$, (4) may be solved by straightforward iteration. The result is

$$N_0(\hat{\nu},\hat{\omega})=N_0{}^0+2\hat{\nu}/3\pi(1-\hat{\omega}^2)^2-[2i\hat{\nu}^2/\pi(1-\hat{\omega}^2)^4]G_2(\hat{\omega})+\ldots, \tag{12}$$

where G_2 is a polynomial of degree six in $\hat{\omega}$.

For the boundary layer $|\hat{\omega}-1|=O(\hat{\nu}^{1/2})$ between resonating and nonresonating particles the expansion parameter is $\hat{\nu}^{1/2}$. The decay of the correlation between wave and particle is no longer exponential as in (11) but algebraic, since for $\mu\sim1$, i.e., $\vec{k}\parallel\vec{v}$, a small deflection in angle does not move the particle out of resonance.

From (11) and (12) we can draw the very important conclusion that the modification of the dielectric constant does not arise from resonance broadening. Replacing, as is frequently done, the real part of N by a Lorentzian, i.e., $\hat{\omega}\to\hat{\omega}+i\Delta\hat{\omega}$, in (10) does not reproduce (12) to any order. [The second term in (12) would require $\Delta\hat{\omega}<0$]. The important modification of $\hat{N}(\hat{\nu},\hat{\omega})$ comes from the $\eta\gg1$ region where $\mathrm{Re}\hat{N}(\hat{\nu},\hat{\omega},\eta)$ goes negative.

The present analysis also does not support earlier contentions[6,2,8] based on estimates of the resonance function from (1), $D=\nu(v)v^2/2$, that the effect of pitch-angle scattering is a cutoff of the linear resonance for $\hat{\nu}(v)\gtrsim1$ and thus a reduction in damping. On the contrary, I find that the weakly collisional, $\hat{\nu}<1$, and the collision-dominated, $\hat{\nu}>1$, regions of velocity space can make contributions of the same order to $\mathrm{Im}\epsilon$, c.f. Fig. 1. The strong velocity dependence of the collisional effects requires in general numerical integration of the dielectric constant (8). A Simpson scheme with adaptive step size was used. One has a continuous transition from Landau damping to collisional damping (or growth in case of a drift $u>\omega/k$) as $1/k\lambda=\nu_e/kv_e$ increases, as shown in Fig. 2. The present problem has a close ana-

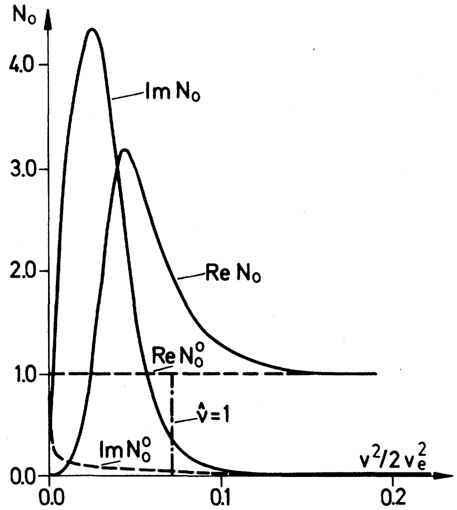

FIG. 1. Resonance function $N_0(\hat{\omega}, \hat{\nu})$ versus $v^2/2v_e^2$; $\hat{\nu}=(1/k\lambda)(v_e/v)^4$, $\hat{\omega}=\omega/kv$, $\omega/kv_e=0.03$, $1/k\lambda=0.02$. Collisionless-theory N_0^0 and its cutoff in earlier theories.

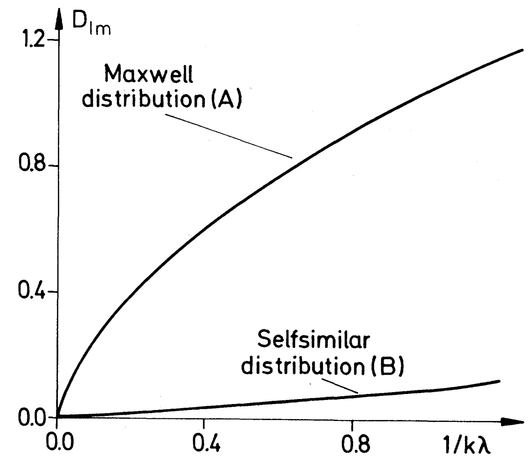

FIG. 2. Collisional modification of dielectric constant. $\mathrm{Im}\epsilon_e=(1+D_{\mathrm{Im}})\,\mathrm{Im}\epsilon_e^0$, $\omega/kv_e=0.03$. $\mathrm{Re}\epsilon_e=(1-D_{\mathrm{Re}})\,\mathrm{Re}\epsilon_e^0$ (not shown), where $D_{\mathrm{Re}}\approx 0.06D_{\mathrm{Im}}$ for curve A, and $D_{\mathrm{Re}}\approx 0.009D_{\mathrm{Im}}$ for curve B, with the same parameter range.

log in neoclassical theory of transport where it also has been shown that a plateau does not exist as the collision frequency decreases.[9]

From Fig. 1 and (8) it follows that collisional effects become much weaker for a distribution which is flat in the low-velocity region. In the turbulent case where quasilinear "flattening" leads to (5), corrections to the quasilinear dielectric constant ϵ^0 are indeed very small. Computer simulation of ion-sound turbulence verifies the validity of quasilinear theory for the wave-electron interaction.[7] To complete the theory for this case I have shown that modified mode-coupling terms (perturbation of collision *operator* by test wave) are also small, using the methods developed here and in an earlier paper.[5]

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