Propagation of Localized Electromagnetic Pulses in Plasmas*

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We study the propagation of three-dimensional, localized, transverse, circularly polarized pulses of electromagnetic radiation in ionized matter. Soliton propagation both below and above the plasma frequency can occur. A critical energy for the formation of solitons exists. The solitons can exist with states of internal excitation.

Recent activity in the field of laser-produced plasmas has prompted considerable interest in the nonlinear propagation of electromagnetic pulses through ionized matter.[1] Effects such as self-focusing[2] and self-induced transparency[3] in plasmas have been analyzed in some detail, although usually for restricted geometries. In this Letter we study the propagation of pulses of finite extent in three dimensions.[4] A self-consistent solution to the coupled nonlinear Maxwell–electron-gas system is obtained. We find qualitatively different features between these pulses and the more familiar one-dimensional solitons. In particular there exists a critical pulse energy below which there can be no propagation. Furthermore we find that there are a set of nonlinear eigenmodes describing states of internal excitation of these pulses. These excited states possess intricate angular and radial structure. The existence of such pulses must have important implications for energy deposition processes in laser-produced plasmas. The modification of the plasma's opacity by the self-induced transparency effect, which permits propagation of pulses whose carrier frequency lies below the plasma frequency, would give rise to an anomalous penetration[5] of laser light into the plasma. This could have an important bearing on the choice of a laser suitable for optimizing the heating of the plasma. Understanding the propagation characteristics of the pulses, i.e., their group velocities and sizes, should be of importance in experimentation with laser-produced plasmas.

Physically the transverse stability of the pulse is brought about through a cancelation of the diffraction effect by the relativistic nonlinearities[6] of the plasma. An analogous situation occurs in the self-focusing problem. The longitudinal stability results from a balance between dispersion forces and nonlinear forces—much as in the phenomenon of one-dimensional dispersive self-induced transparency. Crudely speaking, one may regard the three-dimensional pulses as self-induced-transparency type solitons which have undergone self-focusing to the point where they have achieved a self-consistent pulse shape.

To describe these solutions mathematically let us consider the propagation of a transverse, circularly polarized electromagnetic wave in an ionized gas. Longitudinal excitations (i.e., density fluctuations) will be neglected. We make the local approximation and relate the current density at a given point in space to the field at that point. Furthermore it will be assumed that the thermal velocities of the electrons are small compared with their induced ac velocities. The current density is then given by

$$\vec{J} = -\frac{nea[\hat{\imath}\cos(kz-\omega t+\varphi)+\hat{\jmath}\sin(kz-\omega t+\varphi)]}{[1+(ea/mc^2)^2]^{1/2}}, \tag{1}$$

where n is the electron density and where the vector potential is written as $A = a[\hat{\imath}\cos(kz-wt+\varphi)+\hat{\jmath}\times\sin(kz-wt+\varphi)]$. Included in Eq. (1) are both the nonlinearities due to the velocity-dependent mass effect and the Lorentz force. Insertion of Eq. (1) into the wave equation leads to the complex scalar equation

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\psi e^{i(kz-\omega t)} = \left(\frac{\omega_p}{c}\right)^2 \frac{\psi e^{i(kz-\omega t)}}{(1+|\psi|^2)^{1/2}}, \tag{2}$$

where $\psi = (ea/mc^2)\exp(i\varphi)$ and $\omega_p = (4\pi ne^2/m)^{1/2}$ is the ambient plasma frequency. Much of the following development is motivated by the observation that Eq. (2) is formally Lorentz invariant.

Our interest here will be limited to steady-state pulse propagation and so ψ will be a function of the transverse coordinates and the combination $z - vt$, where v is the group velocity. Under steady-state conditions the phase of ψ must be constant so Eq. (2) becomes

$$\nabla_\perp{}^2\psi + \frac{1}{\gamma^2}\frac{\partial^2\psi}{\partial z^2} + \frac{\omega^2}{\gamma^2 c^2}\psi = \left(\frac{\omega_p}{c}\right)^2 \frac{\psi}{(1+|\psi|^2)^{1/2}}, \qquad (3a)$$

and

$$k\partial\psi/\partial z + (\omega/c^2)\partial\psi/\partial t = 0, \qquad (3b)$$

where we have let $\gamma = [1-(kc/\omega)^2]^{-1/2}$. From Eq. (3b) it follows that $v = kc^2/\omega$. Equation (3a) may be cast in a more symmetric form by introduction of a new variable $z' = \gamma z$. Then we have

$$(\nabla'^2 + q^2)\psi = \left(\frac{\omega_p}{c}\right)^2 \frac{\psi}{(1+|\psi|^2)^{1/2}}, \qquad (4)$$

with $q^2 = (\omega/\gamma c)^2$. Equation (4) is of the form of a nonlinear Schrödinger equation. The reduction of the wave equation to an equivalent eigenvalue problem is similar to that obtained for other nonlinear wave equations.[4] The normalization of ψ may be obtained by looking at pulses of a given energy only. The energy is given by

$$U = \frac{c}{4\pi}\int E^2\,dt\,d^2r_\perp = \frac{\omega^2 q}{4\pi k}\left(\frac{mc}{e}\right)^2 \int |\psi|^2 d^3r'. \qquad (5)$$

To help us conceptualize the solution to the problem we note that ψ is a solution to the Schrödinger equation describing a particle interacting with a potential:

$$[-\tfrac{1}{2}\nabla'^2 + V(\vec{r}\,')]\psi = -\tfrac{1}{2}q^2\psi, \qquad (6)$$

where the potential must satisfy the self-consistency requirement

$$V(\vec{r}\,') = \tfrac{1}{2}[\omega_p/c)^2(1+|\psi(\vec{r}\,')|^2]^{-1/2}. \qquad (7)$$

Pulse-like solutions will exist corresponding to the bound-state solutions of Eq. (6). Hence the bound-state regime is given by $q^2 > 0$ which implies that $v < c$, as is to be expected for the group velocity. The range of interest is thus restricted to $0 \leq q \leq 1$. The asymptotic form of the wave equation will reduce to a linear equation for ψ. Hence we are interested in eigenmodes with the following asymptotic behavior:

$$\psi(\vec{r}\,') \xrightarrow[r'\to\infty]{} (A/r')Y_{lm}(\hat{r}') \exp\{-[(\omega_p/c)^2 - q^2]^{1/2} r'\}. \qquad (8)$$

We further demand that ψ be finite and differentiable for all r'.

It is to be noted that, in general, $V(r')$ will not be spherically symmetric but will only be azimuthally symmetric and invariant under parity reflections. This follows from the insertion of Eq. (8) back into Eq. (7). Thus the spatial structure of ψ can become rather complicated for intermediate values of r'. However, for S states, the spherical symmetry of $V(r')$ is maintained and the equation may be simplified to one dimension. We henceforth restrict our attention to S states and defer the higher L states for future consideration. Let

$$\psi(\vec{r}\,') = u(r')/r'. \qquad (9)$$

Then

$$\frac{d^2u}{dr'^2} + q^2 u = \left(\frac{\omega_p}{c}\right)^2 \frac{u}{[1+(u/r')^2]^{1/2}}, \qquad (10)$$

subject to the boundary conditions $u(0) = 0 = u(\infty)$ and the normalization condition, Eq. (5).

We have not been able to obtain an analytic solution to Eq. (10) and so we resort to numerical integration. The integration procedure proceeds as follows. A value of ω is chosen between 0 and ω_p and k is taken to be zero. A guess for du/dr' at the origin is made and Eq. (10) is numerically integrated out to large r'. The guess is modified until the logarithmic derivative of u matches that obtained from Eq. (8). The integral appearing in Eq. (5) is then evaluated. In this way a series of eigenvalues and eigenvectors were found. The lowest eigenvalue represents the "ground state" of the pulse. The first excited S state represents a radially symmetric solution with one node and higher excited states have sequentially more nodes. The first three S states are shown in Fig. 1. It should be remembered that the pulses are spherically symmetric in the primed coordinate system. In the unprimed (laboratory) system they are distorted into oblate ellipsoids of revolution. The compression factor is given by the Lorentz factor $1/\gamma$. One interesting difference between Eq. (10) and the type of equation one would obtain for one-dimensional solitons is that the nonlinearity here depends on both u and r', whereas there it depends only on the wave function, u. This makes it difficult to find an appropriate integrating factor for the equation. An examination of Eqs. (5) and (10) allows us to deduce

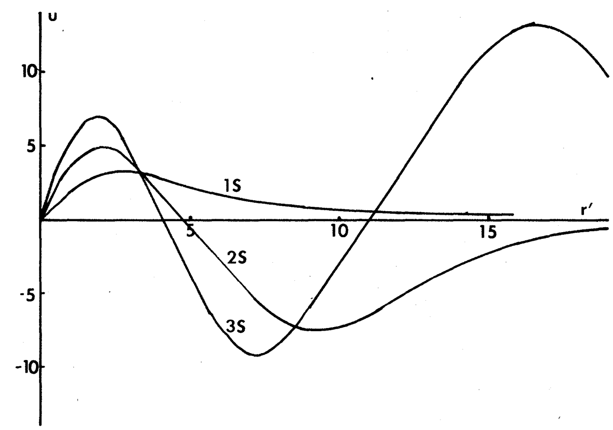

FIG. 1. A plot of the radial part of the pulse amplitude $u(r')$ as a function of the radius, r', both in units of c/ω_p. The first three S states have been denoted by 1S, 2S, and 3S.

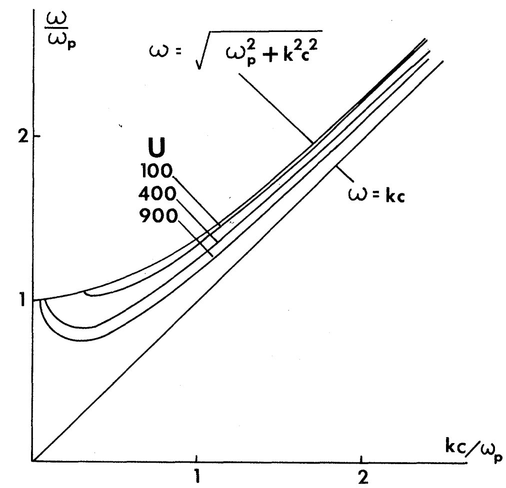

FIG. 2. The isoenergy dispersion curves. The energy is in units of $m^2c^5/\omega_p e^2$.

the following scaling law:

$$U(k_1,\omega_1)/U(k_2,\omega_2)=k_2\omega_1^{\,2}/k_1\omega_2^{\,2}, \tag{11}$$

provided that $\omega_1^{\,2}-k_1^{\,2}=\omega_2^{\,2}-k_2^{\,2}$. This permits us to extrapolate the information learned at $k=0$ to the entire region of the ω-k plane of physical interest,[7] $kc \leq \omega \leq (\omega_p^{\,2}+k^2c^2)^{1/2}$.

In Fig. 2 we present the isoenergy curves associated with the ground state. Two features are noteworthy. The more energetic a pulse is, the further the dispersion curve becomes depressed. This effect has been noted to give rise to the dispersive self-induced transparency phenomenon in one dimension. It is associated with a lowering of the local effective plasma frequency below the ambient plasma frequency by virtue of the variation of the electronic mass with electromagnetic field intensity. Secondly, there exists a critical energy for pulse propagation. This is reminiscent of the self-focusing problem where the concept of a critical power occurs for a beam of finite radius. Here, because of the finite duration of the pulse, however, the existence of a critical power is replaced by that of a critical energy. This is given approximately by

$$U_{\rm cr}\cong 60\,m^2c^5/\omega_p e^2. \tag{12}$$

For example, if we have a plasma generated by an optical laser ($\omega_p \sim 10^{16}$ rad/sec), this would give $U_{\rm cr}\cong 525$ erg. In a purely one-dimensional situation no critical energy or power arises. Because of the infinite transverse extent of the pulse the energy is, in fact, infinite and consequently propagation proceeds for some value of ω and k for arbitrary pulse amplitude.

The origin of the critical energy stems from the self-focusing properties of the nonlinear medium. The critical power associated with the propagation of a finite radius beam is given by[2,3] $P_{\rm cr}=\lambda^2c/(32\pi^2n_2)$. For the relativistic nonlinearity $\epsilon_2=(e\omega_p/mc\omega^2)^2/2$. Hence the critical power is proportional to m^2c^5/e^2 (and terms involving ω/ω_p). Furthermore, in our theory of finite-sized pulses, the typical duration of the pulse is $1/\omega_p$. Thus the origin of the critical energy as given by Eq. (12) can be understood as the combination of the self-focusing properties of the medium and the typical pulse duration.

Let us compute the intensity associated with the ground-state pulse at a frequency corresponding to a 1-μm laser. The intensity is given by

$$I=\frac{cE^2}{8\pi}=\frac{\omega^2m^2c^3}{8\pi e^2}\left(\frac{u}{r'}\right)^2. \tag{13}$$

From Fig. 1 we see that the maximum value of u occurs at $r'\approx 3$ and is $u\approx 3$. Evaluation of the above expression then leads to an intensity of $I=1.4\times10^{18}$ W/cm^2, which is not much different from lasers currently being introduced into experimental laser fusion studies.

In conclusion finite-amplitude, intense, localized, radiation pulses are capable of propagating through plasmas for carrier frequencies both below and above the plasma frequency. The relation between the carrier frequency, the plasma frequency, the group velocity, and the total pulse energy has been presented. A critical pulse energy is found below which no finite propagation is possible.

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[7]To do this let us introduce an auxiliary function $N(k,\omega)=\omega q\int_0^\infty u\,dr'$. It may be readily shown that N obeys the scaling relation

$$N(k,\omega)/N(0,\omega_0)=\omega/\omega_0,$$

so

$$U[k,(k^2+\omega_0^2)^{1/2}]=[(\omega_0^2+k^2)/k\omega_0]N(0,\omega_0),$$

where $\omega_0=(\omega^2-k^2)^{1/2}$. Lines of constant U are thus found from

$$k=[\omega_0U/2N(0,\omega_0)]\pm\{[\omega_0U/2N(0,\omega_0)]^2-\omega_0^2\}^{1/2}.$$

Thus a knowledge of N at a single point determines a line of constant U.

Flute Instabilities in Two-Dimensional Simulations of Strongly Inhomogeneous Theta-Pinch Plasmas

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Particle simulations initialized with inhomogeneous, self-consistent Vlasov equilibria that model theta pinches in the implosion and post-implosion phases have revealed a strongly unstable flute mode for $T_i > T_e$. Frequencies and wavelengths are in good agreement with the linear theory of the lower-hybrid-drift instability. Results agree with quasilinear estimates of heating rates and anomalous resistivity during the initial stages of the instability. The nonlinear saturation of the turbulence is consistent with ion trapping.

Understanding the various instabilities excited in implosion-heating experiments is an important step towards the goal of creating stable high-β plasmas at fusion temperatures. The instabilities, which arise because of the relative drift of electrons and ions caused by gradients in the densities and magnetic field, are responsible for post-implosion anomalous transport effects, such as anomalous magnetic field diffusion. Plasma simulation has become an important tool for studying the basic properties of the instabilities. The Buneman,[1] ion acoustic,[2] electron cyclotron drift,[3] and modified two-stream[4] instabilities have all been explored. These studies, utilizing homogeneous plasmas, included the particle drifts but neglected the effects of the gradients.

In this Letter we report results of two-dimensional particle simulations in θ-pinch-like geometries where steep gradients $[(\nabla n/n)^{-1}\sim(\nabla B/B)^{-1}\sim 0.1c/\omega_{pi}]$ are present. In the hot-ion regime $(T_i>T_e)$, a strong flute instability is observed. The waves grow at roughly half of the lower-hybrid frequency with a wavelength of about 15 electron gyroradii. The instability, which is later identified as the lower-hybrid-drift instability,[5] significantly broadens the sheath region between the plasma and the external magnetic field. The electrons are heated preferentially but not strongly; the electron temperature in the sheath region increases only by a factor of 2. The saturation level of the turbulent electric field is consistent with ion trapping. At saturation the flutes have grown in size to roughly 3 times the original sheath width.

A two-dimensional nonradiative-electromagnetic-particle-in-cell code (DARWIN)[6] has been used