increase in rotation rate to $R^*=1.16$ produced no further qualitative changes, although $P(f)$ broadens substantially.

The transition at $R^*=1$ is sharp, reversible, and nonhysteretic to within a resolution $\delta R^*=0.01$. Although the behavior for $R^*>1$ is independent of the total sample height L, there is some variation with L for $R^*<1$. However, we always detect three basic frequencies (f_1^*, f_2^*, and f_3^*) followed by a sharp onset of aperiodicity.

Our observation of a sharply defined Reynolds number at which the correlation function $C(\tau)$ decays to zero and the discrete peaks in the power spectrum $P(f)$ disappear represents the first clear demonstration that the Landau picture of the onset of turbulence is wrong. The observed behavior seems to be of the general type described by Ruelle and Takens, in which a few nonlinearly coupled modes are sufficient to produce an aperiodic motion. However, there exists no specific theoretical model applicable to this experiment.

Many questions remain unanswered. The arguments of Ruelle and Takens are quite general, and seem to apply to all systems which exhibit normal bifurcations. For these systems how universal is the behavior we have observed? What physical assumptions are inherent in the arguments of Ruelle and Takens? Finally, what is the sequence of events describing the loss of *spatial* correlation of the velocity fluctuations?

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Nonlinear Filamentation of Lower-Hybrid Cones*

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The nonlinear distortion of the propagation cones of lower-hybrid waves is shown to be governed by the modified Korteweg–de Vries equation. Since such an equation admits exact solutions of the multiple-soliton form, it is predicted that filamented cones should be formed when large-amplitude lower-hybrid waves are excited in a plasma.

At the present time it appears that rf heating in the frequency band near the lower-hybrid resonance is one of the promising methods for attaining fusion temperatures in magnetically confined plasmas. A central problem in this heating scheme consists of transporting the rf energy from the outer edge of the plasma to its interior, where it is hoped to be converted into kinetic energy of plasma ions. Various investigations[1-4] of the propagation of lower-hybrid waves have been made in which the linear response of a slightly nonuniform plasma has been emphasized. While this is a sensible method for understanding the basic physics of the problem, it must be realized that linear theory may provide a rather inadequate description of the plasma behavior under actual heating conditions, which by design will involve the application of large rf power levels. In contrast with previous linear studies of this general problem, in this Letter we investigate, analytically, a nonlinear effect which may play an important role in the transport of rf energy to the lower-hybrid layer in a plasma. Specifically, we consider the nonlinear self-distortion of the propagation cones of large-amplitude lower-hybrid waves. The distortion of the propagation cones arises due to the self-consistent modification of the plasma density produced by the ponderomotive force exerted by the lower-hybrid waves on both the electrons and ions.

The geometry of the problem is taken as two dimensional with the z axis along the direction of the external magnetic field and the x axis perpendicular to it. We are interested in describing the spatial dependence of the steady-state potential oscillations $\Phi=\varphi(x,z)\exp(-i\omega t)+\text{c.c.}$ with $\Omega_i \ll \omega \ll \Omega_e$ where Ω_i and Ω_e refer to the ion and electron gyrofrequencies, respectively. The warm-fluid response of the plasma determines the potential amplitude φ through

$$\partial_x(K_\perp\partial_x\varphi)+\partial_z(K_\parallel\partial_z\varphi)+\partial_x^2(a\partial_x^2\varphi)+\partial_z^2(b\partial_z^2\varphi)=0, \qquad (1)$$

where

$$K_\perp=1+(\omega_{pe}/\Omega_e)^2-(\omega_{pi}/\omega)^2,$$
$$K_\parallel=1-(\omega_{pi}/\omega)^2-(\omega_{pe}/\omega)^2,$$
$$a=(\omega_{pi}/\omega)^2(\bar{v}_i/\omega)^2+(\omega_{pe}/\Omega_e)^2(\bar{v}_e/\Omega_e)^2,$$
$$b=(\omega_{pi}/\omega)^2(\bar{v}_i/\omega)^2+(\omega_{pe}/\omega)^2(\bar{v}_e/\omega)^2,$$

ω_{pi} and ω_{pe} refer to the ion and electron plasma frequencies, and $\bar{v}_i$ and $\bar{v}_e$ represent the thermal velocities of the ions and electrons, respectively. In obtaining Eq. (1) only the lowest-order thermal corrections are retained, and the electrons are taken as strongly magnetized while the ions are assumed to behave as unmagnetized for the length scales of interest in this problem. These approximations imply that Eq. (1) should provide an adequate description of phenomena with length scales intermediate between the electron and ion gyroradii.

In the limit in which thermal dispersion is negligible and the external excitation is such that $\omega > \omega_{pi}[1+(\omega_{pe}/\Omega_e)^2]^{-1/2}$, Eq. (1) reduces to

$$\partial_x(K_\perp\partial_x\varphi)-\partial_z(|K_\parallel|\partial_z\varphi)=0. \qquad (2)$$

Furthermore, if the fluctuations have small amplitude so that the nonlinear density changes are negligible and, in addition, if the plasma is uniform, then the exact solutions of Eq. (2) have the functional form $\varphi=\varphi(x-cz)$, where $c=(K_\perp/|K_\parallel|)^{1/2}$. In this limit the problem of rf transport reduces simply to the linear mapping of the rf source along the characteristic lines $x=cz$. These lines are what we loosely refer to as the lower-hybrid cones, since their role is to bound the potential fluctuations within a narrow cone aligned closely to the field lines and emanating from the edges of the rf source. In the following we describe the nonlinear distortions of these cones for a uniform plasma. The results obtained are not expected to be qualitatively different in a nonuniform plasma, except in the immediate vicinity of the lower-hybrid layer near which mode-conversion[1] processes are expected to occur.

The nonlinear distortion of the cones appears in Eq. (1) through the explicit density dependence of the quantities ω_{pi} and ω_{pe}, e.g., $\omega_{pe}=[4\pi e^2 n(x,z)/m]^{1/2}$ with $n(x,z)=n_0+\delta n(x,z)$ where e and m represent the electron charge and mass, and n_0 is the unperturbed density. The nonlinear density changes δn therefore modify $K_\perp$ and $K_\parallel$, which in turn produce the distortion of the linear characteristics. In particular these distortions may take the form of filaments along which intense localized electric fields appear. These fields can interact strongly with the plasma particles in the region far from the lower-hybrid layer; hence this effect may seriously alter the transport of rf energy to the interior of the plasma.

The ponderomotive force $\vec{F}_j$ acting on the jth charged species can be obtained by averaging the term $\vec{v}\cdot\nabla\vec{v}$, appearing in the appropriate fluid equation, over one cycle of the lower-hybrid oscillations (considered here to be rapidly varying compared to the slow density changes). This procedure yields

$$\vec{F}_j=-\hat{x}\partial_x\left[\left(\frac{\omega_{pj}\omega}{\omega^2-\Omega_j^2}\right)^2\frac{|\partial_x\varphi|^2}{8\pi}+\frac{\omega_{pj}^2}{\omega^2-\Omega_j^2}\frac{|\partial_z\varphi|^2}{8\pi}\right]-\hat{z}\partial_z\left[\left(\frac{\omega_{pj}}{\omega}\right)^2\frac{|\partial_z\varphi|^2}{8\pi}+\frac{\omega_{pj}^2}{\omega^2-\Omega_j^2}\frac{|\partial_x\varphi|^2}{8\pi}\right]. \qquad (3)$$

Equation (3) shows that the ponderomotive force pushes both the electrons and the ions across as well as along the magnetic field lines. The push in the x direction produces a slow nonlinear drift which in a uniform plasma does not give rise to density changes, hence it plays no role in the present investigation. However, the z component in Eq. (3) can indeed give rise to density cavities aligned along the magnetic field. These cavities can significantly modify the linear characteristics of the lower-hybrid oscillations. The corresponding density of electrons, n_-, and of ions, n_+, can be obtained from the adiabatic response to the ponderomotive force along the field lines, i.e.,

$$n_-(x,z) = n_0 \exp\left\{-\frac{1}{T_e}\left[-e\varphi_s + \left(\frac{\omega_{pe}}{\omega}\right)^2 \frac{|\partial_z\varphi|^2}{8\pi n_0} - \left(\frac{\omega_{pe}}{\Omega_e}\right)^2 \frac{|\partial_x\varphi|^2}{8\pi n_0}\right]\right\},$$
$$n_+(x,z) = n_0 \exp\left\{-\frac{1}{T_i}\left[e\varphi_s + \left(\frac{\omega_{pi}}{\omega}\right)^2 \frac{|\nabla\varphi|^2}{8\pi n_0}\right]\right\}, \quad (4)$$

where T_i and T_e refer to the ion and electron temperatures, respectively. In Eq. (4), φ_s represents an ambipolar potential which must be self-consistently determined by the requirement that $n_+ \simeq n_-$. This procedure yields, for small nonlinear density changes, the familiar expression encountered in the study of nonlinear effects in unmagnetized plasmas, namely

$$\frac{\delta n}{n_0} = -\frac{|\nabla\varphi|^2}{8\pi n_0 T}, \quad T = T_e + T_i. \quad (5)$$

In arriving at Eq. (5) from Eq. (4) use is made of the zeroth-order (i.e., cold-plasma) dispersion relation for the lower-hybrid oscillations.

One can now insert Eq. (5) into Eq. (1) to obtain the following nonlinear equation for φ

$$K_{\perp 0}\partial_x^2\varphi - |K_{\parallel 0}|\partial_z^2\varphi + a_0\partial_x^4\varphi + b_0\partial_z^4\varphi - \alpha_0\partial_x\left(\frac{(\nabla\varphi)^2}{8\pi n_0 T}\partial_x\varphi\right) + \beta_0\partial_z\left(\frac{(\nabla\varphi)^2}{8\pi n_0 T}\partial_z\varphi\right) = 0, \quad (6)$$

where

$$\alpha = (\omega_{pe}/\Omega_e)^2 - (\omega_{pi}/\omega)^2, \quad \beta = (\omega_{pe}/\omega)^2 + (\omega_{pi}/\omega)^2,$$

and the zero subscripts indicate unperturbed values. Since we are interested in extracting an exact analytic formulation of the filamentation problem, we limit ourselves to small but yet finite wave amplitudes, so that the balancing between the nonlinear distortion and the thermal dispersion in Eq. (6) can be treated consistently as the first-order correction to the linear propagation cones of a cold plasma. The spirit of the calculation consists of introducing the distortion through the functional form $\varphi = \varphi(y,\tau)$ with $y = x - cz$ and $\tau = cz$. The dependence of φ on τ is considered to provide a correction of order ϵ ($\epsilon \ll 1$), and the thermal dispersion as well as the nonlinearity are thought to make contributions also of order ϵ. The mathematical procedure then consists of retaining in Eq. (6) only those terms of order ϵ and disregarding those of order ϵ^2 and higher. This implies that since $\partial_x\varphi = \partial_y\varphi$ and $\partial_z\varphi = -c\partial_y\varphi + c\partial_\tau\varphi$, then in calculating the higher order derivatives with respect to z one can neglect the terms $\partial_\tau^l\varphi$ having $l > 1$. A further simplification can be implemented by noticing that $c \ll 1$, hence in Eq. (6) $(\nabla\varphi)^2 \simeq (\partial_x\varphi)^2$ with high accuracy. Carrying out the straightforward algebra yields

$$2\partial_\tau u + \frac{3u^2}{8\pi n_0 T}\left(\frac{c^2\beta - \alpha}{K_\perp}\right)_0 \partial_y u + \left(\frac{bc^4 + a}{K_\perp}\right)_0 \partial_y^3 u = 0, \quad u = \partial_y\varphi. \quad (7)$$

Then introducing the scaled variables

$$\xi = [3(c^2\beta_0 - \alpha_0)/(b_0c_0^4 + a_0)]^{1/2}(x - cz),$$
$$t = [3(c^2\beta_0 - \alpha_0)]^{3/2}(b_0c_0^4 + a_0)^{-1/2}(2K_{\perp 0})^{-1}cz, \qquad v = (8\pi n_0 T)^{-1/2}\partial_y\varphi, \quad (8)$$

one finally arrives at the basic nonlinear equation which governs the filamentation of the lower-hybrid cones, namely

$$\partial_\tau v + v^2\partial_\xi v + \partial_\xi^3 v = 0, \quad (9)$$

which is known in the literature[5,6] as the "modified Korteweg-de Vries equation," and is one of the few nonlinear partial differential equations for which exact solutions are known. In particular, it has been shown by Hirota[5] that Eq. (9) has exact solutions consisting of N solitons (N being an arbitrary integer). Therefore, the present analysis predicts that the external excitation of large-amplitude lower-hybrid oscillations should lead to the generation of multiple filaments along which intense electric fields of soliton shape are formed. Of course, the number of filaments formed as well as their amplitude and width are determined by the particular geometry and conditions used for exciting the oscillations. In a recent experiment,[7] in which large-amplitude lower-hybrid oscillations were excited by an antenna inside the plasma, Gekelman and Stenzel have observed that intense localized electric fields and associated density cavities, of the type considered here, are formed along the cones in regions of the plasma far from the lower-hybrid layer.

A particularly simple solution of Eq. (9) which illustrates the filamentation and self-focusing of the lower-hybrid cones is the two-soliton solution[5] given by

$$\tan\hat{\varphi}(\xi,t)=(e^{r_1}+e^{r_2})\left[1-\left(\frac{P_1-P_2}{P_1+P_2}\right)^2 e^{r_1+r_2}\right]^{-1}, \quad \partial_\xi\hat{\varphi}=v, \tag{10}$$

where $r_j=P_j\xi-P_j^3t-g_j$, and in which P_j and g_j are the parameters which determine the amplitude and path of the jth soliton. The nonlinear characteristics followed by these solitons make an angle θ_j with the z axis given by $\tan\theta_j=c(1+\lambda P_j^2)$, $\lambda=3(c^2\beta_0-\alpha_0)/(2K_{\perp 0})$, thus showing that the effect of the nonlinearity is to cause an amplitude-dependent increase in the angle of the lower-hybrid cones. Since, in general, the amplitude of the electric field will be different along different filaments, it follows that two isolated filaments will intersect (i.e., focus) at some point in the xz plane, thus leading to the formation of highly localized and intense electric fields. Equation (1) predicts also the existence of spatial oscillations in the intersection between filaments (i.e., recursive scattering of the filaments) when the condition $P_1=P_2^*$ is attained.

In summary, the present work has shown that the self-consistent distortion of the lower-hybrid cones is governed by the modified Korteweg-de Vries equation. Physically, this equation describes the balance between the filamentation tendency produced by the density changes caused by the ponderomotive force along the field lines and the spreading or diffraction due to the thermal pressure. The resulting effect is that intense localized electric fields of the soliton form are generated. The interaction of the plasma particles with these fields may alter the transport of energy to the interior of the plasma, as has been observed in a recent small-scale experiment.

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