Two-Dimensional and Three-Dimensional Envelope Solitons

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We have obtained symmetric envelope solitons in two and three dimensions in a medium with a saturated nonlinear susceptibility given by $\chi_{\rm NL} \sim \exp(-|\phi|^2)-1$, where ϕ is an appropriately normalized field amplitude. Examples for an electromagnetic wave and an electrostatic wave trapped in a plasma are presented.

Extensive investigations have recently been carried out for one-dimensional propagation of envelope solitons.[1] They are obtained as stationary localized solutions of the nonlinear Schrödinger equation by balancing the effect of group dispersion, $d^2\omega/dk^2$, and that of quadratic nonlinearity in the susceptibility, $\chi_{\rm NL} \sim |\phi|^2$, where ϕ is an appropriately normalized field amplitude. However, when the equation is generalized to two and three dimensions with the same quadratic nonlinearity, localized solutions are found to be nonstationary except for a very special case of measure zero; they tend to either collapse or expand indefinitely depending on the initial condition.[2] The question then arises as to whether higher-order nonlinearity can stop the collapse or expansion and admit localized stationary solutions. In this paper, we present such solutions for the case when the nonlinear susceptibility has the form $\chi_{\rm NL} \sim \exp(-|\phi|^2)-1$. This form of nonlinear susceptibility arises in the problem of steady propagation of large-amplitude high-frequency waves in a plasma. Physically, the nonlinearity arises in this case through a local density depression created by the ponderomotive force of the wave.[3] The nonlinearity saturates when most of the plasma particles are expelled from regions of high field intensity. In this Letter, we present examples for both electromagnetic and electrostatic waves confined in a stationary plasma cavity. Although we specifically discuss only the problem of waves in a plasma, the results obtained are quite general and may be regarded as general soliton-like solutions for a certain class of nonlinear wave equations.

We first consider steady propagation (in the z direction) of an azimuthally polarized electromagnetic wave which is localized in the radial direction. Let ϕ be the modulus of the azimuthal electric field normalized as $\phi = e|E_\theta|/\omega[m(T_e+T_i)]^{1/2}$, where the notation used is standard. If we normalize the radial distance by the collisionless skin depth, c/ω_p, where ω_p is the plasma frequency in the absence of the wave, the stationary wave equation takes the following form:

$$\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}r\phi + \left[1-\sigma-\exp(-\phi^2)\right]\phi = 0, \qquad (1)$$

where σ is the normalized frequency mismatch from the plane-wave dispersion relation, $\sigma = (\omega_p^2 + c^2k^2 - \omega^2)/\omega_p^2$, and k is the wave number in the z direction. We have solved this equation as an eigenvalue problem (for eigenvalue σ) for the case of the lowest eigenvalue (ground-state solution). From the structure of the equation, we can infer the solution to be of the form $I_1(\sigma^{1/2}r)$ near the origin and $K_1(\sigma^{1/2}r)$ at large distances, these being smoothly connected near the maximum with the form $J_1([1-\sigma-\exp(-\phi_0^2)]^{1/2}r)$, where ϕ_0 is the maximum value of ϕ. There also exist oscillatory solutions corresponding to higher eigenvalues, but we shall not consider them here, because they have a greater extent in the radial direction and hence are quite likely to be unstable against smaller-size perturbations.[4] Numerical results for the solutions are shown in Fig. 1. We see from this figure that for σ exceeding 0.1, ϕ_0 becomes of order or greater than unity, so that the approximation of quadratic nonlinearity breaks down.

Two features are to be observed in the results. First, ϕ_0 increases monotonically with σ, and secondly, the position of the maximum amplitude, $r = r_m$, as well as the width of the radial profile decrease as σ increases up to about 0.3 and then starts increasing indefinitely as σ approaches unity, with a broad minimum which extends from

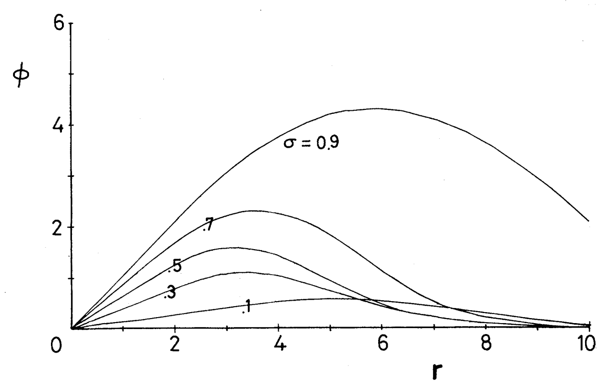

FIG. 1. Two-dimensional solitons for azimuthally polarized electromagnetic waves for $\sigma = 0.1$, 0.3, 0.5, 0.7, and 0.9.

$\sigma \sim 0.3$ to $\sigma \sim 0.6$. The minimum value of r_m as well as the minimum width at half-intensity maximum ($\phi = \phi_0/\sqrt{2}$) are about three times the collisionless skin depth.

The first feature originates from the fact that as the amplitude increases, the susceptibility, and hence the index of refraction, deviate from the unperturbed value, and so does the frequency mismatch σ. The second feature may be interpreted as follows. For small σ, or for small ϕ_0, the main part of the solution at large r is governed by $K_1(\sigma^{1/2}r)$ which shrinks toward the origin as σ increases. This simply represents the tendency of collapse with increasing nonlinearity. On the other hand, for large σ, ϕ becomes substantial over a wide range of r; then neglecting $\exp(-\phi^2)$, one obtains a solution of the form $J_1((1-\sigma)^{1/2}r)$ which expands indefinitely as σ approaches unity. Physically, this represents the fact that as ϕ increases, the index of refraction approaches unity (because the plasma is expelled), and hence the solution becomes close to a free-space plane wave which has an infinite width. The solution with the minimum width represents the situation where the collapsing tendency balances the tendency to expand because of expulsion of plasma particles.

So far we have considered only a radial localization of the wave. Such a solution is, however, likely to be unstable against self-modulation of the amplitude along the z direction.[5] Several mechanisms[6,7] have been proposed which may cause such an instability in a plasma. However, as long as $\omega \gg \omega_p$ and the light wave is untrapped in the z direction, the modulation propagates with a very high speed, so that even a slight inhomogeneity of plasma parameters along the z direction can lead to a low-level saturation of the instability because of convective effects.[7] A deleterious situation may occur only when ω is close to ω_p and the light wave is trapped in a locally underdense region. The electromagnetic filament will then be split into small pieces in the z direction and the final stationary state will become a three-dimensional soliton localized in all directions. Restricting ourselves to azimuthally polarized electromagnetic waves, we can obtain such a solution by replacing k^2c^2/ω_p^2 in σ of Eq. (1) by $-\partial^2/\partial z^2$, where z is normalized by the skin depth. The equation then becomes

$$\left[\frac{\partial^2}{\partial z^2} + \frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}r\right]\phi + \{1-\sigma-\exp(-\phi^2)\}\phi = 0\,, \quad (2)$$

where σ now stands for $(\omega_p^2 - \omega^2)/\omega_p^2$. The forms of the soliton-like solution of this equation in the region $r \gtrsim r_m$ for two limiting cases, $\phi^2 \gg 1$ and $\phi^2 \ll 1$, are

$$\phi \sim \begin{cases} \cos qz\, J_1((1-\sigma-q^2)^{1/2}r) & (\phi^2 \gg 1) \\ \exp(-\gamma|z|)K_1((\sigma-\gamma^2)^{1/2}r) & (\phi^2 \ll 1), \end{cases} \quad (3)$$

where q and γ are certain positive parameters with $(\gamma^2 + q^2) < 1$ and σ is assumed to have a value between γ^2 and $1-q^2$. The condition that these two solutions be connected smoothly gives a relation between q and γ. One can easily infer that they are of the same order. However, since we are now dealing with a partial differential equation, we have two independent eigenvalues, q and σ. This brings in an arbitrariness in the shape of the soliton. Now, for a given wave energy, $I = \int_{-\infty}^{\infty} dz \int^{\infty} r\,dr |\phi|^2$, the amplitude ϕ_0^2 has a maximum at a certain value of eccentricity in the r-z directions. For instance, if $\phi^2 \gg 1$ in the main region of soliton, $\phi_0^2 \sim Iq(1-\sigma-q^2)$ which assumes a maximum value at $q = [(1-\sigma)/3]^{1/2}$. This solution may represent the most stable state. The typical size of this solution is again given by the collisionless skin depth. If we further maximize ϕ_0^2 by varying σ, we obtain $\sigma = 0$. This corresponds to an electromagnetic soliton in which the field oscillates at the plasma frequency and might represent the final saturation state of the induced Brillouin backscattering at the cutoff density.

We now consider the electrostatic case. As in Ref. 2, we treat a spherically symmetric electron-plasma oscillation confined in a stationary plasma cavity. We now take ϕ to be the modulus of the radial derivative of the electrostatic po-

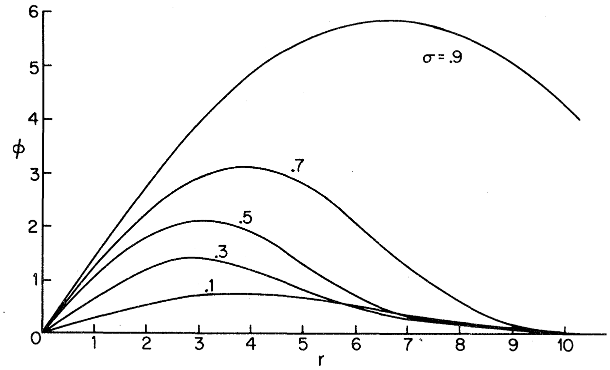

FIG. 2. Soliton profile for spherically symmetric electrostatic wave for $\sigma = 0.1$, 0.3, 0.5, 0.7, and 0.9.

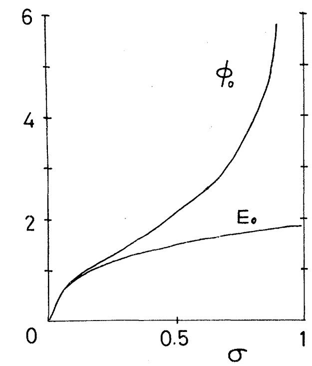

FIG. 3. Relation between σ and ϕ_0 or $E_0 = \omega\phi_0/\omega_p$ for spherically symmetric electrostatic solitons.

tential $\varphi(r)$ normalized as $\phi = e|\partial\varphi/\partial r|\omega_p/\omega[3T_e \times (T_e + T_i)]^{1/2}$, where the radial distance is now normalized by $3^{1/2}\lambda_D$, λ_D being the Debye length $[(T_e/m\omega_p)^{1/2}]$. Our equation then becomes[2]

$$\frac{\partial}{\partial r}\frac{1}{r^2}\frac{\partial}{\partial r}r^2\phi + \left[1 - \sigma - \exp(-\phi^2)\right]\phi = 0, \qquad (4)$$

where $\sigma = (\omega_p^2 - \omega^2)/\omega_p^2$. Localized solutions of this eigenvalue equation have recently been obtained by Wilcox and Wilcox for the case $\sigma = \frac{1}{2}$.[8] The "ground-state" solution now takes the form $I_{3/2}(\sigma^{1/2}r)/\sqrt{r}$ near the origin and $K_{3/2}(\sigma^{1/2}r)/\sqrt{r}$ for large r, as shown in Fig. 2. Although the maximum amplitude, ϕ_0, depends on σ in a way similar to the electromagnetic case, there is one distinct difference from the latter; namely, the maximum electric field amplitude, $E_0 \equiv \omega\phi_0/\omega_p$, now saturates to a finite value as σ approaches unity, since ω vanishes for $\sigma \to 1$ (Fig. 3). Note that in the electromagnetic case $\omega \to \omega_p$ for $\sigma \to 1$, so that there is no qualitative difference in the behaviors of E_0 and ϕ_0. The dependence of the maximum position and the width on σ is similar to the electromagnetic case. The minimum width in this case becomes of order $5\lambda_D$ which occurs at $\sigma \sim 0.3$. The solution corresponding to this minimum size will most likely be stable against all perturbations described by the same electrostatic-wave equation.

In order to apply the above results to realistic plasma situations, one has to take into account various dissipation mechanisms as well as possible instabilities due to coupling of electromagnetic and electrostatic components. For electromagnetic solitons, perturbations in the azimuthal direction produce coupling with electrostatic waves. Since the minimum size of the electromagnetic soliton is much greater than the Debye length, this coupling may cause an efficient conversion from electromagnetic to electrostatic waves, and the electromagnetic soliton may eventually be split up into many electrostatic solitons of smaller size. Now, the minimum radius of the electrostatic soliton is of order $5\lambda_D$. Such a localized oscillation will suffer a strong transit-time damping due to thermal electrons, as has been demonstrated by the one-dimensional simulation.[9] However, for the case of spherical solitons, the transit-time damping will be reduced as compared with the one-dimensional case, since most of the particles pass sideways through the soliton and do not experience the acceleration by the maximum field, ϕ_0. A preliminary calculation using the Fokker-Planck–equation model for the distribution function shows that the damping rate is reduced by a factor 20–40 in the spherical case as compared with the one-dimensional case for the same typical wavelength.[10] Even with this reduction factor one can no longer neglect the transit-time damping if the radius becomes as small as $5\lambda_D$. In this case, we need to include the effect of a pump which balances the dissipation. We, however, believe that there exist many situations where the inclusion of both pump and dissipation does not drastically change the overall structure of our soliton solutions.

There are many other effects which we have ignored in the above argument. These include the effect of ions trapped in the cavity, that of electrons reflected by the ponderomotive force, long-range interactions among solitons via particles and untrapped waves, possible rotation of the soliton, and so on. These problems are now under investigation and will be discussed elsewhere.

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[1]See, for instance, V. I. Karpman, *Nonlinear Waves in Dispersive Media* (Nauka, Moscow, 1973), Chap. 5.

[2]V. E. Zakharov, Zh. Eksp. Teor. Fiz. 62, 1745 (1972) [Sov. Phys. JETP 35, 908 (1972)].

[3]P. Kaw, G. Schmidt, and T. Wilcox, Phys. Fluids 16, 1522 (1973).

[4]J. Denavit, N. R. Pereira, and R. N. Sudan, Phys. Rev. Lett. 33, 1435 (1974).

[5]Dynamical properties of an equation similar to Eq. (1) [but the term $(\partial/\partial r)r^{-1}(\partial/\partial r)\phi$ being replaced by $r^{-1}(\partial/\partial r)r(\partial/\partial r)\phi$] was studied numerically by J. H. Marburger and E. Dawes, Phys. Rev. Lett. 21, 556 (1968), and V. E. Zakharov *et al.*, Zh. Eksp. Teor. Fiz. 60, 136 (1971) [Sov. Phys. JETP 33, 77 (1971)]. See, also, V. E. Zakharov, Zh. Eksp. Teor. Fiz. 53, 1735 (1967) [Sov. Phys. JETP 26, 994 (1968)] for linear instability.

[6]C. E. Max, J. Arons, and B. Langdon, Phys. Rev. Lett. 33, 209 (1974).

[7]K. Mima and K. Nishikawa, to be published.

[8]J. Z. Wilcox and T. J. Wilcox, Bull. Amer. Phys. Soc. 19, 861 (1974).

[9]E. J. Valeo and W. L. Kruer, Phys. Rev. Lett. 33, 750 (1974).

[10]A. Ishida and K. Nishikawa, to be published.

Method of Generating Very Intense Positive-Ion Beams

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The combination of multiply reflected electrons and positive ion flow in a reflex triode arrangement is analyzed. Under certain conditions it is possible to generate very intense beams of positive ions with this device. The analysis demonstrates that the energy loss and scattering of the electrons as they pass through the anode have a major effect on the ion and electron currents. Solid fractional-range anodes are shown to produce more intense ion beams than semitransparent mesh anodes.

Both laser and electron beams[1,2] are possible energy sources for the heating and compression required to release fusion energy from dense, inertially contained plasmas. The feasibility of using intense beams of positive ions for this purpose has also been investigated.[3,4] The main problem is that the current density of the available ion beams is far too low. However, there is no inherent technological limitation on ion-beam intensity, and we propose here a method of producing short pulses of MeV ions at current densities of ~10 kA/cm^2.

Recently, Humphries, Lee, and Sudan[5] have produced intense beams of positive ions with the "reflex-triode" or "double-diode" configuration shown in Fig. 1. Multiple transits of electrons are used to increase the ratio of ion current to electron current. The principal disadvantage of the ion beams measured so far is that the current densities are low (20–50 A/cm^2). In connection with experiments[6,7] employing the double-diode configuration of Fig. 1, Smith has pointed out that the use of a foil anode, instead of the

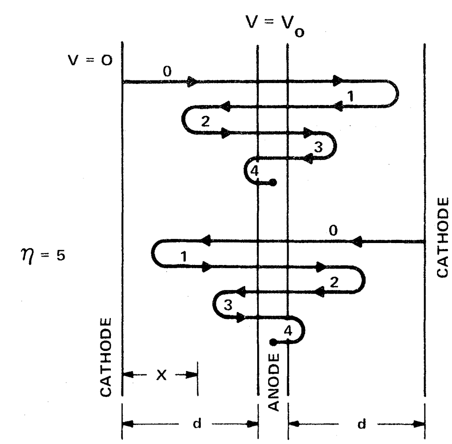

FIG. 1. Multiple reflection of electrons in a "reflex triode." A fractional-range anode is located between two cathodes so that the electrons make many transits before stopping. Positive ions flow from the anode plasma to both cathodes.