Laboratories in checking some of our source materials for radioactive contamination.

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Exact Classical Solution for the 't Hooft Monopole and the Julia-Zee Dyon*

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We present an exact solution to the nonlinear field equations which describe a classical excitation possessing magnetic and electric charge. This solution has finite energy and exhibits explicitly those properties which have previously been found by numerical analysis.

Recently 't Hooft[1] has proposed a model for a magnetic monopole which arises as a static solution of the classical equations for the SU(2) Yang-Mills field coupled to an SU(2) Higgs field. The model has been extended by Julia and Zee[2] so that the monopole becomes a dyon, possessing both electric and magnetic charge. The purpose of this note is to present an exact analytic solution for a particular version of these models.

The Lagrangian density for the model is[3]

$$\mathcal{L} = -\tfrac{1}{4}F^{\mu\nu a}F_{\mu\nu}{}^{a} - \tfrac{1}{2}\Pi^{\mu a}\Pi_{\mu}{}^{a} + \tfrac{1}{2}\mu^2\varphi^a\varphi^a - \tfrac{1}{4}\lambda(\varphi^a\varphi^a)^2, \tag{1}$$

where

$$F_{\mu\nu}{}^{a} = \partial_\mu A_\nu{}^a - \partial_\nu A_\mu{}^a + e\epsilon^{abc}A_\mu{}^b A_\nu{}^c, \tag{2}$$

and

$$\Pi_\mu{}^a = \partial_\mu\varphi^a + e\epsilon^{abc}A_\mu{}^b\varphi^c. \tag{3}$$

The field equations in the static limit where all time derivatives are zero are

$$\partial_i F^{\mu i a} + e\epsilon^{abc}A_i{}^b F^{\mu i c} = e\epsilon^{abc}\Pi^{\mu b}\varphi^c, \tag{4}$$

and

$$\partial_i \Pi^{ia} + e\epsilon^{abc} A_\mu{}^b \Pi^{\mu c} = -\mu^2 \varphi^a + \lambda(\varphi^b \varphi^b)\varphi^c. \tag{5}$$

The Wu-Yang[4]-'t Hooft–Julia-Zee *Ansatz* is to seek a solution of the form

$$A_i{}^a = \epsilon_{aij}\hat{r}_j[1-K(r)]/er, \tag{6}$$

$$A_0{}^a = \hat{r}_a J(r)/er, \tag{7}$$

$$\varphi^a = \hat{r}_a H(r)/er. \tag{8}$$

Fields which satisfy Eqs. (4) and (5) also produce an extremum of the canonical Hamiltonian obtained from (1):

$$\mathcal{H} = \int d^3r[\tfrac{1}{4}F_{ij}{}^a F_{ij}{}^a - \tfrac{1}{2}F_{0i}{}^a F_{0i}{}^a + \tfrac{1}{2}\Pi_i{}^a\Pi_i{}^a - \tfrac{1}{2}\Pi_0{}^a\Pi_0{}^a - \tfrac{1}{2}\mu^2\varphi^a\varphi^a + \tfrac{1}{4}\lambda(\varphi^a\varphi^a)^2]. \tag{9}$$

When expressed in terms of $K(r)$, $J(r)$, and $H(r)$ this becomes

$$\mathcal{H} = \frac{4\pi}{e^2}\int_0^\infty dr\left(K'^2 + \frac{(K^2-1)^2}{2r^2} - \frac{J^2K^2}{r^2} - \frac{(rJ'-J)^2}{2r^2} + \frac{H^2K^2}{r^2} + \frac{(rH'-H)^2}{2r^2} - \frac{\mu^2H^2}{2} + \frac{\lambda H^4}{4e^2r^2}\right), \tag{10}$$

and the field equations reduce to

$$r^2K'' = K(K^2-1) + K(H^2-J^2), \tag{11}$$

$$r^2J'' = 2JK^2, \tag{12}$$

$$r^2H'' = 2HK^2 + (\lambda/e^2)(H^3 - C^2r^2H), \tag{13}$$

where $C = \mu e/\sqrt{\lambda}$.

We seek solutions such that the square of the Higgs field goes to a constant as $r\to\infty$. Then from Eq. (13) we see that $H/r \xrightarrow[r\to\infty]{} \pm C$. For definiteness we choose the positive sign. The particular version of these equations that we will consider is to take $\lambda\to 0$ with C fixed. Then the requirement $H/r \xrightarrow[r\to\infty]{} C$ is no longer forced upon us. Nevertheless, we shall look for solutions of Eqs. (11)–(13) for which it continues to be valid.

It is easy to verify that a solution of Eqs. (11)–(13) with $\lambda = 0$ and regular boundary conditions at $r=0$ and ∞ is given by

$$K = Cr/\sinh(Cr), \tag{14}$$

$$J = 0, \tag{15}$$

$$H = Cr\coth(Cr) - 1, \tag{16}$$

which therefore constitutes an exact solution of 't Hooft's model for λ and μ^2 both set equal to 0.

Furthermore, by choosing

$$K = Cr/\sinh(Cr), \tag{17}$$

$$J = \sinh\gamma[Cr\coth(Cr) - 1], \tag{18}$$

$$H = \cosh\gamma[Cr\coth(Cr) - 1], \tag{19}$$

we have a solution of our version of the model of Julia and Zee with $H/r \xrightarrow[r\to\infty]{} C\cosh\gamma$. Here γ is an arbitrary constant. Clearly the solution of 't Hooft's model corresponds to $\gamma = 0$.

The reader may wonder how such a solution was discovered. The answer is that if one seeks trial functions with good boundary conditions for a variational calculation, the hyperbolic sine and tangent arise quite naturally. Substitution of a few trial functions of this type in the equations revealed that they were almost satisfied. With a bit of "shimmying" the expressions (14)–(16) emerged.

The exact solutions leave unchanged most of the conclusions which have been deduced from the numerical analyses. Specifically, let us investigate the magnetic monopole strength $4\pi g$, the electric charge Q, and the mass M of the excitation we have found.

The Abelian electromagnetic field has been identified by 't Hooft[1,5] as

$$\mathfrak{F}_{\mu\nu} = \partial_\mu(\hat{\varphi}^a A_\nu{}^a) - \partial_\nu(\hat{\varphi}^a A_\mu{}^a) - (1/e)\epsilon^{abc}\hat{\varphi}^a\partial_\mu\hat{\varphi}^b\partial_\nu\hat{\varphi}^c, \tag{20}$$

where $\hat{\varphi}^a = \varphi^a(\varphi^b\varphi^b)^{1/2}$. Using Eqs. (6)–(8) we have

$$\mathcal{B}_i = \tfrac{1}{2}\epsilon_{ijk}\mathfrak{F}_{jk} = -\hat{r}_i/er^2, \tag{21}$$

and

$$\mathcal{E}_i = -\mathfrak{F}_{0i} = \hat{r}_i\frac{d}{dr}\left(\frac{J}{r}\right) = \hat{r}_i\sinh\gamma\left(\frac{1-K^2}{er^2}\right). \tag{22}$$

We see that our solution represents a point monopole of strength $4\pi g = -4\pi/e$. In addition, since $K \to 1$ as $r \to 0$ there is a cloud of electric charge of amount $Q = (4\pi/e)\sinh\gamma$ with no pointlike core. Thus the solution represents a point monopole surrounded by a cloud of electric charge.

The energy or mass of the solution can be calculated from the Lagrangian density written in generally covariant form as $M = \int d^3r\, T^{00}(\vec{r})$, where

$$T^{\mu\nu} = (2/\sqrt{-g})\partial(\mathcal{L}\sqrt{-g})/\partial g_{\mu\nu} = F^{\mu\lambda a}F_\lambda{}^{\nu a} + \Pi^{\mu a}\Pi^{\nu a} + g^{\mu\nu}\mathcal{L}. \tag{23}$$

We find the same expression for M as for $\mathcal{H}$ given by Eq. (10) except that the two terms with negative signs have those signs reversed. For the case $\lambda = 0$, Eqs. (12) and (13) allow us to write

$$\frac{1}{2r^2}(rJ' - J)^2 + \frac{J^2K^2}{r^2} = \frac{1}{2}\frac{d[J(J' - J/r)]}{dr}, \tag{24}$$

with a corresponding equation for the H terms so that

$$M = (4\pi/e^2)\{\tfrac{1}{2}[H(H' - H/r) + J(J' - J/r)]|_0^\infty + \int_0^\infty dr[K'^2 + \tfrac{1}{2}(K^2 - 1)^2/r^2]\}. \tag{25}$$

The integral is easily evaluated with the result

$$M = (C/\alpha)\cosh^2\gamma, \tag{26}$$

where $\alpha = e^2/4\pi$. The constant C governs the falloff of the Yang-Mills field as $r \to \infty$ and is thus identified with its mass.

A comparison with the numerical results of 't Hooft and Julia and Zee is instructive. For $\gamma = 0$ 't Hooft finds $M = (C/\alpha)f(\lambda/e^2)$, where f varies from 1.1 for $\lambda/e^2 = 0.1$ to 1.44 for $\lambda/e^2 = 10$. Julia and Zee compute $f = 1.42$ for $\lambda/e^2 = 0.5$. We find $f = 1$ for $\lambda/e^2 = 0$.

Secondly, Julia and Zee find for $\lambda/e^2 = 0.5$ solutions with $Q = 0.324e/\alpha$ which correspond to $\gamma = 0.319$ and 1.038, respectively. For such choices of γ we would find $M = 1.10C/\alpha$ and $2.56C/\alpha$, respectively, while they find $M = 2.62C/\alpha$ and $2.86C/\alpha$, respectively.

A few concluding remarks concerning stability are in order. Our solution appears to be unstable against changes in C since there is a continuum of solutions, each with mass proportional to C. But C is the only mass parameter in the theory and so it sets the scale of length. Thus solutions for different values of $C \neq 0$ are identical with respect to the appropriate length scale and hence can be considered stable, at least with regard to changes in C.

Concerning changes in γ it would appear on the classical level that since the mass depends continuously on γ, only $\gamma = 0$ can lead to stability unless some classical reason can be found to quantize $Q = (e/\alpha)\sinh\gamma$.

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