angular variables. At low temperature, we have essentially $\xi \ll \Delta$ which makes $\beta E \simeq \beta\Delta + \beta\xi^2/2\Delta$ so that ξ scales like $T^{1/2}$. From Eq. (4) we obtain a factor $(T^{1/2})^3/T = T^{1/2}$, but we must take into account that we have only a probability $\exp[-\Delta/T]$ of finding a scattering quasiparticle. Finally, we obtain $\tau_{CE} \sim T^{-1/2}e^{\Delta/T}$. This behavior actually agrees with the low-temperature behavior of the relaxation time[5] for viscosity and thermal conductivity. This is not surprising since this behavior could be deduced from the same scaling arguments. In the same way, we expect the viscosity and thermal-conductivity relaxation times to behave like $1/T^4$ in the A phase. (Naturally, to obtain the viscosity, for example, one has to take into account other factors than the relaxation time, but their temperature dependence can be easily deduced from a relaxation-time approximation.[10])

Finally, it is remarked that when the gap is completely established, say $T/T_c \sim 0.7$–0.8, the spin-conserving character of the normal quasiparticle collisions is no longer felt so that in this range τ_{CE} should be of order of a typical relaxation time at T_c.

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[7]P. Bhattacharrya, C. J. Pethick, and H. Smith, following Letter [Phys. Rev. Lett. 35, 473 (1975)] have also derived this result together with the exact coefficient by solving the kinetic equation (1).

[8]Note that, although τ_{CE} is diverging, $\omega_0\tau_{CE} \ll 1$ when T goes to T_c since the longitudinal frequency ω_0 goes to zero like $(1-T/T_c)^{1/2}$. For the transverse linewidth, one could have situations where the hydrodynamic condition $\omega\tau_{CE} \ll 1$ is no longer satisfied. But this would require such high magnetic fields and temperature so near T_c that the linewidth would be very small anyway and any effect would be difficult to observe.

[9]I am very grateful to O. Valls for pointing out this fact to me.

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Spin Relaxation in Superfluid ^{3}He†

P. Bhattacharyya
Department of Physics and Materials Research Laboratory, University of Illinois, Urbana, Illinois 61801

and

C. J. Pethick
Department of Physics and Materials Research Laboratory, University of Illinois, Urbana, Illinois 61801, and Nordita, Copenhagen, Denmark

and

H. Smith
Physics Laboratory I, H. C. Ørsted Institute, University of Copenhagen, DK-2100 Copenhagen Ø, Denmark

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We show that near the transition temperature T_c the relaxation time introduced in Leggett and Takagi's phenomenological theory of spin relaxation in superfluid ^{3}He is equal to the relaxation time of a normal-state quasiparticle at the Fermi energy at T_c, and is independent of the superfluid state. Combescot and Ebisawa's relaxation time is found to diverge as $(T_c - T)^{-1/2}$. These results are obtained by deriving and solving exactly the Boltzmann equation for quasiparticles in the superfluid.

The authors of two recent Letters[1,2] in which spin relaxation in superfluid Fermi liquids is treated phenomenologically arrive at different conclusions about NMR linewidths close to the transition temperature T_c. The differing results reflect differences in the assumptions made about the rate of relaxation processes. In this Comment we discuss the problem of spin relaxation from a microscopic point of view. We derive the quasiparticle Boltzmann equation valid for temperatures close to T_c, and then solve it exactly to calculate the rate of magnetic relaxation processes. We obtain microscopic expressions for the relaxation times introduced by Leggett and Takagi[1] ($\tau_{\rm LT}$) and by Combescot and Ebisawa[2] ($\tau_{\rm CE}$). $\tau_{\rm LT}$ is found to be equal to the relaxation time of a normal-state quasiparticle at the Fermi energy at T_c and *does not depend on the particular state of the superfluid.* The latter result could not have been anticipated on the basis of phenomenological considerations. $\tau_{\rm CE}$ is found to vary as $(T_c-T)^{-1/2}$, in agreement with the work of Ambegaokar[3] and Combescot[4] but in disagreement with the assumption made in Ref. 2. Since $\tau_{\rm LT}$ can be determined directly from NMR measurements, such as the ringing down of the wall-pinned mode in B-^{3}He[5-7] this gives one a direct measurement of the quasiparticle relaxation time at the Fermi energy, a quantity which cannot be directly measured in any other experiment. This promises to provide us with valuable new information about quasiparticle scattering amplitudes in the normal state.

The total spin polarization, S, of a superfluid may be expressed as the sum of two parts[1]: the spin from Cooper pairs, S_p, and the spin of the quasiparticles, S_q. The first is due to the deviation of the superfluid coherence factors from their values in the unpolarized state, and the second is due to the deviation of the quasiparticle distribution from its value in the unpolarized state. Consider now a situation in which S_q is different from its equilibrium value for the particular value of S the system has. If the system is homogeneous, the Boltzmann equation for the distribution of quasiparticles in the superfluid, $n_{\vec{p}\sigma}$, is

$$\partial n_{\vec{p}\sigma}/\partial t = I\{n_{\vec{p}\sigma}\}, \tag{1}$$

where $I\{n_{\vec{p}\sigma}\}$ is the collision integral. Let us consider the experimentally interesting case of longitudinal relaxation,[1] in which the deviation of the quasiparticle spin from its equilibrium value is in the same direction as the total spin, which we take to be the z direction. The most convenient quasiparticle states to use are ones in which the z component of spin is diagonal. Let us denote the single-quasiparticle states in the absence of any spin polarization by $|\vec{p}^+\rangle$ and $|\vec{p}^-\rangle$ respectively. Then

$$\langle\vec{p}^+|\sigma_z|\vec{p}^+\rangle = -\langle\vec{p}^-|\sigma_z|\vec{p}^-\rangle \equiv s_{\vec{p}},$$
$$\langle\vec{p}^+|\sigma_z|\vec{p}^-\rangle = 0. \tag{2}$$

Here $s_{\vec{p}}$ is not in general equal to 1 since σ_z also has matrix elements between states for which the numbers of quasiparticles differ by two. (Note that real transitions between such states are not important in the low-frequency limit $\hbar\omega \ll \Delta$, where Δ is the superfluid gap.) These eigenstates are particularly convenient since the distribution function, which is a matrix in general, is diagonal when written in terms of them.

Let us assume that S_q is varying for some reason. As we shall see, at temperatures close to T_c, the characteristic time scale for changes in S_q is long compared with the characteristic time for the quasiparticle distribution to relax to the local equilibrium distribution (for a given value of S_q), which is of the order of the normal-state collision time. Thus on the left-hand side of the Boltzmann equation we may insert a local equilibrium distribution given by

$$n_{\vec{p}\sigma}{}^0 = \{\exp[(E_{\vec{p}} - \tfrac{1}{2}\hbar\sigma s_{\vec{p}}H)/k_{\rm B}T]+1\}^{-1}. \tag{3}$$

H, the field acting on the quasiparticles, is essentially the difference between the up- and down-spin chemical potentials,[1] and $E_{\vec{p}}$ is the quasiparticle energy in the superfluid. The quasiparticle spin is given by

$$S_q = \sum_{\vec{p}\sigma}(\hbar/2)\sigma s_{\vec{p}} n_{\vec{p}\sigma}, \tag{4}$$

where $\sigma = \pm 1$ is the spin index. In local thermodynamic equilibrium and for small H, Eq. (4) reduces to $S_q = \gamma^{-2}\chi_{q0}H$, where

$$\chi_{q0} = \sum_{\vec{p}\sigma}\gamma^2 s_{\vec{p}}{}^2(\hbar/2)^2(-\partial n_{\vec{p}}{}^0/\partial E_{\vec{p}}) \tag{5}$$

is the quasiparticle susceptibility[1] without Fermi-liquid effects and γ is the gyromagnetic ratio. The left-hand side of the Boltzmann equation may therefore be written as $-\sigma s_{\vec{p}}(\hbar/2)(\partial n_{\vec{p}}{}^0/\partial E_{\vec{p}}) \times\gamma^2\chi_{q0}{}^{-1}(\partial S_q/\partial t)$.

Let us now turn to the collision term, which we shall calculate only to first order in Δ. In the superfluid the number of quasiparticles is not conserved. One therefore has to take into account not only processes in which two quasiparticles scatter but also those in which one quasiparticle decays into three or three coalesce to produce one. Processes in which four excitations are created or destroyed to not contribute to leading order in Δ. The linearized collision term in the Boltzmann equation including scattering, coalescence, and decay processes may be derived by methods quite analogous to those we used in calculating the viscosity,[8] and is

$$\partial \delta n_1/\partial t = -\sum_{2,3,4} W_N(1,2;3,4) n_1^0 n_2^0(1-n_3^0)(1-n_4^0)\delta_{\vec{p}_1+\vec{p}_2,\vec{p}_3+\vec{p}_4} \times \delta(E_1+E_2-E_3-E_4)[\psi_1 + s_1(s_2\psi_2 - s_3\psi_3 - s_4\psi_4)], \tag{6}$$

where the indices 1–4 refer to momentum and spin variables, the deviation function is defined by $n_i^0(1-n_i^0)\psi_i = n_i - n_i^0$, and W_N is the normal-state transition probability. To obtain (6) we used the fact that for the situation considered here, $\psi_{\vec{p}\sigma} = -\psi_{\vec{p}-\sigma}$. It is convenient to introduce a modified deviation function

$$Q \equiv -\psi_{\vec{p}\sigma}[\sigma(\hbar/2)\tau_0 s_{\vec{p}} 2\cosh(x/2)(\partial S_q/\partial t)/\chi_{q0}]^{-1},$$

where τ_0 ($\propto T^{-2}$) is the characteristic normal-state quasiparticle relaxation time, defined in Jensen, Smith, and Wilkins,[9] and x is $\xi_{\vec{p}}/k_B T$, $\xi_{\vec{p}}$ being the normal-state quasiparticle energy measured from the Fermi energy. Generally Q is anisotropic because of the anisotropy of the superfluid state. (In both the Anderson-Brinkman-Morel (ABM) and Balian-Werthamer (BW) states, for instance, $s_{\vec{p}}$ depends on $\hat{p}$.) However, the angular-dependent part of Q may be shown to be of order $\Delta/k_B T_c$ times the spherically symmetric part, and therefore Q may be taken to be spherically symmetric. By manipulations similar to those used in calculating the viscosity,[8] one finds the following equation for Q:

$$\frac{1}{2\cosh(x/2)} = \frac{x^2+\pi^2}{2} Q(x) - \int_{-\infty}^{+\infty} dx' \frac{(x-x')}{2\sinh\frac{1}{2}(x-x')} \int \frac{d\Omega_{\hat{p}'}}{4\pi} s^2(x',\hat{p}') Q(x'). \tag{7}$$

The effective spin of a quasiparticle, $s_{\vec{p}}$, is very nearly unity except for quasiparticles with energies $\lesssim \Delta(\hat{p})$. Only collisions involving these quasiparticles violate quasiparticle-spin conservation, and convert quasiparticle spin into the spin of Cooper pairs. Equation (7) is essentially the same as the equation for the deviation function in the case of viscosity, but with a few important differences. First, the equation in the case of viscosity has a parameter $\alpha/2$,[9] which is a ratio of two different angular averages of W_N, multiplying the integral term. Here the parameter is identically equal to unity since quasiparticle spin is conserved in the normal state. Second, in the viscosity equation the quasiparticle velocity takes the place of the effective quasiparticle spin, $s_{\vec{p}}$. As a consequence of quasiparticle-spin conservation in the normal state, the solution to Eq. (7) is singular in the normal limit ($s_{\vec{p}} \to 1$), since $Q(x) \propto 1/\cosh(x/2)$ is a solution of the homogeneous equation. Contrary to what one might at first sight expect, this singular behavior makes the equation trivial to solve, and to leading order in $\Delta/k_B T_c$,

$$Q(x) = \frac{2}{\pi^2\lambda} \frac{1}{2\cosh(x/2)}. \tag{8}$$

This may easily be demonstrated by expanding Q in terms of the eigenfunctions used in normal-state transport calculations,[9] and showing that only the lowest eigenfunction is important. The parameter λ, introduced by Leggett and Takagi,[1] is the Cooper-pair susceptibility, χ_{C0}, relative to the total susceptibility, neglecting Fermi-liquid effects, and to leading order in Δ is

$$\int (d\Omega_{\hat{p}}/4\pi) \int_{-\infty}^{\infty} dx(1 - s^2(x,\hat{p}))(-\partial n^0/\partial x) \propto (T_c - T)^{1/2}.$$

Calculating the deviation from local equilibrium $n_{\vec{p}\sigma} - n_{\vec{p}\sigma}^0$ and inserting it into Eq. (4) to determine the quasiparticle spin S_q, one finds

$$\sum_{\vec{p}\sigma} \sigma s_{\vec{p}}(\hbar/2)(n_{\vec{p}\sigma} - n_{\vec{p}\sigma}^0) = S_q - \gamma^{-2}\chi_{q0} H = -\frac{2\tau_0}{\pi^2\lambda}\frac{\partial S_q}{\partial t} \tag{9a}$$

or

$$\frac{\partial S_q}{\partial t} = -\frac{\lambda}{\tau(0)}(S_q - \gamma^{-2}\chi_{q0}H) = -\frac{1}{\tau_{CE}}(S_q - \gamma^{-2}\chi_{q0}H), \tag{9b}$$

where $\tau(0) = 2\tau_0/\pi^2$ is the relaxation time of a normal-state quasiparticle at the Fermi energy.[9] Equation (9b) is easily obtained by calculating $(\partial S_q/\partial t)$ from Combescot and Ebisawa's kinetic equation.[2] Thus $\tau_{CE} = \tau(0)/\lambda \propto (T_c - T)^{-1/2}$.

To identify Leggett and Takagi's relaxation time, we note that since the structure of the Cooper pairs adjusts to the field in a time $\sim(\hbar/\Delta) \ll \tau_{CE}$, their spin polarization S_p $(=S-S_q)$ is given by its equilibrium value, $\gamma^{-2}\chi_{C0}H$, for the field H, whence $H=\gamma^2\chi_{C0}^{-1}(S-S_q)$. Using the relation $\chi_{q0}/\chi_{C0}=(1-\lambda)/\lambda$, we can rewrite Eqs. (9) in the form

$$\frac{\partial S_q}{\partial t}=-\frac{1}{\tau(0)}(S_q-S_{q0}), \tag{10}$$

where $S_{q0}=(1-\lambda)S$. Leggett and Takagi's basic equation is for $\eta=S_p-S_{p0}$ $[=-(S_q-S_{q0})]$, where $S_{p0}(S)=\lambda S$ is the equilibrium value of S_p. From Eq. (10), using the fact that $\partial S/\partial t=R_D$, the dipolar torque,[1] one finds that

$$\partial\eta/\partial t=(1-\lambda)R_D-\eta/\tau(0), \tag{11}$$

which is identical with Leggett and Takagi's Eq. (7), apart from the replacement of their phenomenological relaxation time τ_{LT} by $\tau(0)$. Since $\tau(0)$ is finite at T_c, it is obvious that the temperature dependence of physical quantities such as the cw resonance linewidths, and relaxation rates in pulsed NMR and nonlinear longitudinal ringing experiments agree precisely with those obtained by Leggett and Takagi.[1] Note also that the relation $\tau_{CE}=\tau_{LT}/\lambda$ is valid for arbitrary temperatures and not only near T_c, where $\tau_{LT}=\tau(0)$. That τ_{LT} is independent of the superfluid state is a result of the particular form of the collision integral in the superfluid, and, as we mentioned above, could not have been anticipated on the basis of phenomenological calculations. It is physically reasonable that $\tau(0)$ determines the spin relaxation rate, since as we have seen, only collisions involving quasiparticles with energies $\lesssim\Delta(\ll k_BT)$ violate quasiparticle-spin conservation.

The calculations above have a number of implications for experimental work. First, the lack of dependence of τ_{LT} on the superfluid state should by checked by making measurements in the A and B phases near the polycritical point. Second, the fact that spin-relaxation measurements determine $\tau(0)$ gives one a consistency relation between a number of experimental measurements. The normal-state viscosity depends on the quasiparticle scattering amplitude through the parameters $\tau(0)$ and α. α may be determined directly from measurements of the relative change in the viscosity in the superfluid state close to T_c, and hence the normal-state viscosity can now be calculated in terms of experimentally measured quantities, independent of any assumption about the angular dependence of the normal-state quasiparticle scattering amplitude. We plan to give a more detailed comparison of our results with experiment in a future publication. Apart from giving one a better understanding of the superfluid phases of ^{3}He, further measurements of spin relaxation will also be of considerable value in helping one to understand quasiparticle scattering in normal ^{3}He.

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