Kinetic Process of Plasma Heating Due to Alfvén Wave Excitation

Akira Hasegawa and Lui Chen*
Bell Laboratories, Murray Hill, New Jersey 07974
(Received 4 September 1974)

We elucidate the kinetic process of plasma heating by the resonant excitation of the shear Alfvén wave. The heating occurs as a result of the damping of the modified (by the finite ion Larmor radius and electron inertia) Alfvén wave which is excited by mode conversion at the resonant layer. The heating rates of ions and of electrons are comparable in the collisional regime; otherwise electrons are predominantly heated.

It has been shown recently[1] that the singular property of the shear Alfvén wave, $\omega^2 = k_\parallel^2 v_A^2(x)$, in a nonuniform plasma can be utilized to heat a plasma by resonant absorption. However, because the ideal magnetohydrodynamic (MHD) equations were used, the detailed absorption mechanism was not clear. Here, we present a result of kinetic theory and show that (1) the incoming perturbation is converted at the resonant layer to the modified (by the finite ion Larmor radius and electron inertia) Alfvén wave that propagates across the magnetic field, quite analogous to the case of the Bernstein wave,[2] (2) the modified Alfvén wave is dissipated by ions as a result of viscous damping, as well as by electrons as a result of collisional or Landau damping, and (3) the total absorption rate is approximately the same as the MHD result[1] if $\nu r/\omega\rho_i \gtrsim O(1)$, where ν and ω are the effective damping rate and the angular frequency of the wave, r is the plasma radius, and ρ_i is the average ion Larmor radius.

We also show that the applied frequency $\omega = k_\parallel v_A$ can be chosen such that it becomes comparable to the bounce frequency ω_b of electrons trapped in the local mirror, and hence the heating can eliminate or reduce the number of marginally trapped electrons. The scheme should, therefore, be quite attractive in presenting a possibility of eliminating the two fundamental problems of the tokamak: heating and enhanced diffusion due to banana-orbit effects.

We first discuss the absorption rate. For this purpose, we derive the wave equation of the shear Alfvén wave with finite ion Larmor radius and electron inertia in a nonuniform plasma. We use the drift kinetic equation for electrons and the Vlasov equation for ions. For field quantities we use two-field components, a scalar potential φ [to describe the perpendicular (with respect to the magnetic field) electric field $\vec{E}_\perp = -\nabla_\perp\varphi$], and a parallel electric field $E_\parallel$. $E_\parallel$ is introduced here because in this low-frequency regime electron dynamics is mainly in the parallel direction and the parallel conductivity of electrons with finite inertia is finite. These variables conveniently decouple the compressional magnetic field perturbation $B_\parallel \propto \nabla_\perp \times \vec{E}_\perp$.

For simplicity, we employ a local planar geometry in which x is the direction of nonuniformity which is perpendicular to the dc magnetic field; z is parallel to the magnetic field. In the toroidal system, x, y, and z correspond to radial, poloidal, and toroidal coordinates, respectively. For the frequency range $\omega \ll \omega_{ci}$ (the ion cyclotron frequency), quasi neutrality can be assumed. The field equations for $\varphi(x)\exp[i(k_\perp y + k_\parallel z - \omega t)]$ and $E_\parallel(x)\exp[i(k_\perp y + k_\parallel z - \omega t)]$ become

$$n_i - n_e = 0, \tag{1}$$

$$(d^2/dx^2 - k_\perp^2)(ik_\parallel\varphi + E_\parallel) = -i\omega\mu_0 J_\parallel. \tag{2}$$

Equation (2) comes from $[\nabla\times(\nabla\times\vec{E})]_\parallel = \mu_0\,\partial J_\parallel/\partial t$. We first consider a regime in which $k_x\rho_i < 1$. For a low-β plasma with $m_e/m_i \ll \beta \ll 1$, the number-density perturbations for ions and electrons, n_i and n_e, and the current-density perturbation in the parallel direction, $J_\parallel$, are obtained from the Vlasov equation by use of the operator technique,[3]

$$\frac{en_i}{\epsilon_0} = \frac{\omega_{pi}^2}{\omega_{ci}^2}\frac{d}{dx}\left(1 + \frac{3}{4}\rho_i^2(1 - i\delta_i)\frac{d^2}{dx^2}\right)g\frac{d\varphi}{dx}, \tag{3}$$

$$-en_e/\epsilon_0 = -i(\omega_{pe}^2/k_\parallel^2 v_{Te}^2)k_\parallel g E_\parallel(1 + i\delta_e), \tag{4}$$

$$J_\parallel = -i\omega\epsilon_0(\omega_{pe}^2/k_\parallel^2 v_{Te}^2)(1 + i\delta_e)gE_\parallel, \tag{5}$$

where v_{Te} is the electron thermal speed ($> v_A$ for $\beta > m_e/m_i$), $g(x)$ is a nonuniform density profile normalized to unity at the maximum-density point[3] as shown in Fig. 1, and δ (>0) is the fractional dissipation rate of the wave due to the viscous damping ($\delta_i \sim 14\nu_{ii}/15\omega$),[4] collisional damping ($\delta_e \sim \nu_{ei}v_A/k_\parallel v_{Te}^2$), or Landau damping ($\delta_e \sim \omega/k_\parallel v_{Te}$). In deriving these expressions, we have assumed $k_\parallel(\sim R^{-1}) \ll k_\perp(\sim r^{-1}) \ll |\partial/\partial x|(\sim\rho_i^{-1})$, and have kept the lowest-order corrections due to

$\rho_i^2\,\partial^2/\partial x^2$, where R is the major toroidal radius. These assumptions, which are easily justified for a tokamak plasma, conveniently decouple the ion acoustic wave, the drift wave, and the magnetosonic wave from the Alfvén wave considered here.

The wave equation can be constructed by substituting Eqs. (3)–(5) into (1) and (2). Inside the resonant points $x=\pm x_0$, where $\omega^2=k_\parallel^2 v_A^2(\pm x_0)$ (i.e., $|x|<x_0$ in Fig. 1), we ignore $k_\perp$ with respect to d/dx because $d/dx \sim \rho_i^{-1}$; then the equation becomes

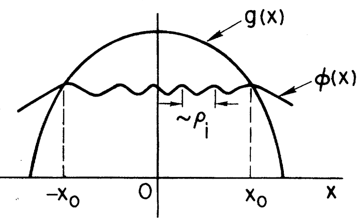

FIG. 1. Density profile $g(x)$ and schematic profile of wave potential $\varphi(x)$.

$$\frac{d}{dx}\left[\left(\frac{\omega^2}{k_\parallel^2 v_A^2}g-1\right)\frac{d\varphi}{dx}+\left(\frac{\omega^2}{k_\parallel^2 v_A^2}\frac{3}{4}\rho_i^2(1-i\delta_i)\frac{d^2}{dx^2}+\frac{d}{dx}\frac{1}{g}\frac{T_e}{T_i}\rho_i^2(1-i\delta_e)\frac{d}{dx}\right)g\frac{d\varphi}{dx}\right]=0, \qquad (6)$$

where v_A is the Alfvén speed at the maximum-density point, and T_e and T_i are the electron and ion temperatures. If we note the proportionality between φ and ξ_x ($\vec{\xi}$ is the plasma displacement vector), when $\rho_i \to 0$, this wave equation reduces to the one obtained by use of the ideal MHD equations,[5] i.e.,

$$\frac{d}{dx}\epsilon(x)\frac{d\varphi}{dx}=0, \qquad (6')$$

where

$$\epsilon(x)=\omega^2/k_\parallel^2 v_A^2(x)-1. \qquad (7)$$

Equation (6) shows that on the higher-density side of the resonant points $\pm x_0$ [$\epsilon(\pm x_0)=0$], there exists a propagating wave. In a uniform plasma, the modified Alfvén wave has the following dispersion relation:

$$\omega^2=k_\parallel^2 v_A^2\{1+[\tfrac{3}{4}(1-i\delta_i)+(T_e/T_i)(1-i\delta_e)]\rho_i^2 k_x^2\}. \qquad (8)$$

If $\delta \to 0^+$, k_x is real when $v_A(x)$ is smaller than $v_A(\pm x_0)$. Therefore, for a nonuniform plasma with a finite extent in the x direction, the wave may be trapped at the central region of the plasma, similar to the case of the Bernstein wave.[2] Consequently, to calculate the absorption rate one can no longer depend on the resonant absorption that originates from the logarithmic singularity of Eq. (6′), but one must solve Eq. (6).

Since the problem is analogous to the case of the Bernstein wave (or, more precisely, Buchsbaum-Hasegawa resonance), one can use the result obtained for that case.[6] If we assume $(dg/dx)/g \ll \rho_i^{-1}$, Eq. (6) can be solved by connecting the WKB solutions in the regions $|x|>x_0$ and $|x|<x_0$. The solution for $|x|>x_0$ is then

$$\varphi=C[\mathrm{P}\int_0^x dx\,[\epsilon(x)]^{-1}+(\pi/\kappa)\cot(\theta-\tfrac{1}{4}\pi)], \qquad (9)$$

where

$$\kappa=\partial\epsilon/\partial x\,|x_0 \sim r^{-1},$$

and

$$\theta=\int_0^{x_0}k_x(x)\,dx\equiv\theta_R+i\theta_I. \qquad (10)$$

$k_x(x)$ is the nonuniform wave number (complex) given by

$$k_x^2(x)=\frac{1}{\rho_i^2}\frac{\omega^2/k_\parallel^2 v_A^2(x)-1}{\frac{3}{4}(1-i\delta_i)+(T_e/T_i)(1-i\delta_e)}. \qquad (11)$$

While, the solution of Eq. (6′), which is obtained by use of the ideal MHD equations, reads

$$\varphi_{\mathrm{MHD}}=C[\mathrm{P}\int_0^x dx\,[\epsilon(x)]^{-1}-i\pi/\kappa]. \qquad (9')$$

As was shown in Ref. 1, the absorption rate of the applied field energy is proportional to the imaginary part of φ in Eqs. (9) and (9′). With the correction for finite ion Larmor radius, if $x_0 \sim \rho_i$, the imaginary part has a sharp structure because of the cotangent function [Eq. (9)]. However, as in the case of tokamaks, where $x_0 \gg \rho_i(\sim 10^{-2}x_0$ to $10^{-3}x_0)$, if $\theta_I \simeq \bar{\delta}x_0/\rho_i \gtrsim O(1)$, the cotangent function reaches $-i$, where $\bar{\delta}=(4\delta_i/3+T_e\delta_e/T_i)/2(\frac{3}{4}+T_e/T_i)^{3/2}$. Thus, we recover the MHD result, Eq. (9′). The absorption rate per cycle is given by $(B_x^2/\mu_0)\kappa/k_\perp$, where B_x is the wave magnetic field at $x=x_0$.

Let us now discuss the heating rate of different species. Because the absorption is localized only near $x=x_0$, whereas the heating is more uniformly distributed, we should distinguish them carefully. To study this problem, we need the dispersion relation for $k_\perp \rho_i > 1$. In this regime, Eq. (3) is modified to

$$en_i/\epsilon_0 = -(\omega_{pi}^2/v_{Ti}^2)(1 - i\delta_i')\varphi, \tag{3'}$$

where[4] $\delta_i' = [3(\pi+1)/8\sqrt{\pi}](\nu_{ii}/\omega)k_x\rho_i$, and the corresponding dispersion relation is given by

$$\omega^2 = k_\parallel^2 v_A^2 k_x^2 \rho_i^2 \left(\frac{T_e/T_i}{1+i\delta_e} + \frac{1}{1+i\delta_i'}\right). \tag{8'}$$

By comparing Eqs. (8) and (8′), we see that the two dispersion relations connect rather smoothly. The heating rate of ions, $n_0 dT_i/dt$, is then obtained as

$$n_0 dT_i/dt = \tfrac{1}{2}\mathrm{Re}(\vec{J}\cdot\vec{E})_{\rm ion} = \mathrm{Re}[(\omega/k_x)en_iE_x^*]$$
$$= \begin{cases} 0.7\nu_{ii}(k_x\rho_i)^2(\epsilon_0|E_x|^2/2)\omega_{pi}^2/\omega_{ci}^2 \text{ for } k_x\rho_i < 1, \\ 0.9\nu_{ii}(k_x\rho_i)^{-1}(\epsilon_0|E_x|^2/2)\omega_{pi}^2/\omega_{ci}^2 \text{ for } k_x\rho_i > 1. \end{cases} \tag{12}$$

Here we note that near $x = x_0$, $(\epsilon_0|E_x|^2/2)\omega_{pi}^2/\omega_{ci}^2 = B_y^2/2\mu_0$ and also that because $\partial\omega/\partial k_x \propto (k_x\rho_i)^{-1}$, the local heating rate remains approximately constant, and is given by $\nu_{ii}B_y^2(x_0)/2\mu_0$.

The heating rate of electrons is obtained from Eqs. (1) and (5),

$$n_0 dT_e/dt = \tfrac{1}{2}\mathrm{Re}(J_\parallel E_\parallel^*) = \omega\delta_e(\omega_{pe}^2/k_\parallel^2 v_{Te}^2)\epsilon_0|E_\parallel|^2/2$$
$$= \begin{cases} \omega\delta_e(k_x\rho_i)^2(T_e/T_i)(\epsilon_0|E_x|^2/2)\omega_{pi}^2/\omega_{ci}^2 \text{ for } k_x\rho_i < 1, \\ \omega\delta_e(k_x\rho_i)^{-2}(T_e/T_i)(\epsilon_0|E_x|^2/2)\omega_{pi}^2/\omega_{ci}^2 \text{ for } k_x\rho_i > 1. \end{cases} \tag{13}$$

The fractional electron damping rate δ_e is given by

$$\delta_e = \begin{cases} (\nu_{ei}/k_\parallel v_{Te})v_A/v_{Te} \text{ for collisional damping,} \\ \alpha\omega/k_\parallel v_{Te} + \alpha'(\omega/\omega_b)^3 \text{ for Landau damping,}^7 \end{cases}$$

where α is a reduction factor ~0.1 which designates the circulating electrons, and α' is $n_{\rm trapped}/n_{\rm total}$.

For tokamak parameters, we see that the heating rates of electrons and ions are approximately the same if the collisional damping dominates. If we assume a machine with a toroidal field of 50 kG, a plasma density of 10^{14} cm^{-3}, and a major radius R of 10 m, $\omega = v_A/R$ becomes ~10^6 sec^{-1}. Hence, if $T_i \lesssim 1$ keV, then $\nu_{ii} \gtrsim 3\times10^3$ sec^{-1} and ion heating due to the direct dissipation of the wave may be expected. However, if $T_i > 1$ keV, the dissipation due to the electron Landau damping dominates, and the ions may be heated by ion-electron collisions. To deposit energy of the order of 1 MJ/m^3 in 1 sec into the plasma, the amplitude of the modulated magnetic field is obtained from $B_y^2/2\mu_0 = 10^6/\omega\bar{\delta} \sim 10^3$; thus $B_y \sim 500$ G, where we took the average $\bar{\delta}$ to be 10^{-3}. Note that B_y obtained above is the y component of the wave magnetic field after the mode conversion. B_y can be shown from Eq. (6) to be enhanced and is related to B_x at $x = x_0$ through $B_y = (1/k_y)(\kappa/\rho_i)^{1/2}B_x$. If we use this relation and note that the area of absorption is given by $L_yL_z(\mathrm{Im}k_x)^{-1}$, the present result for the heating rate $\sim\omega\delta B_y^2L_yL_z(\mathrm{Im}k_x)^{-1}$ can be reduced to the absorption rate obtained by the MHD result which is $\omega B_x^2L_yL_zk_y^{-1}\kappa/k_y$. Namely, the applied magnetic field, B_x, can be much smaller than B_y and is on the order of 15 G.

For the tokamaks, we propose to use also $\omega \sim \langle\omega_b\rangle$ so that one may reduce the number of trapped electrons which will eventually reduce the diffusion rate.[8] Such a choice is possible because[9] $\langle\omega_b\rangle \sim v_{Te}(r/R)^{1/2}/Rq$, $q = rB_0/RB_\theta$, whereas $v_{Te}/v_A = (m_iT_e/m_eT_i)^{1/2}\beta^{1/2} \gtrsim 1$ and $k_\parallel \sim R^{-1}$.

We would like to thank Professor Roy W. Gould for calling our attention to the importance of mode conversion of the resonant layer, and Professor David E. Baldwin, Professor Marshall N. Rosenbluth, and Professor Dieter J. Sigmar for valuable discussions and comments on the manuscript.

*Present address: Plasma Physics Laboratory, Princeton University, Princeton, N. J. 08540.

[1]A. Hasegawa and L. Chen, Phys. Rev. Lett. 32, 454 (1974); L. Chen and A. Hasegawa, Phys. Fluids 17, 1399 (1974); H. Grad, Phys. Today 22, No. 12, 34 (1969); W. Grossmann and J. Tataronis, Z. Phys. 261,

203 (1973).
[2]S. J. Buchsbaum and A. Hasegawa, Phys. Rev. Lett. 12, 685 (1964), and Phys. Rev. 143, 303 (1966).
[3]A. Hasegawa, Phys. Fluids 8, 761 (1965). We assume here that $\omega(=k_{\parallel}v_A) >> \omega_e^*(=k_y v_{Te}^2|\nabla \ln g|/\omega_{ce})$ so that the drift wave is decoupled. This is justified if $\rho_i k_y << 1$ for a low-β plasma with $\beta \sim (r/R)^2$. Thus, the equilibrium distribution $g(x+v_y/\omega_c)f(v^2)$ can be approximated as $g(x)f(v^2)$.
[4]A. A. Rukhadze and V. P. Silin, Usp. Fiz. Nauk 96, 87 (1968) [Sov. Phys. Usp. 11, 659 (1969)].
[5]Equation (6′) is obtained by using $\varphi \propto \xi_x$ and substituting $k_y \rightarrow 0$ in Eq. (10) of Hasegawa and Chen, Ref. 1. This leads to the logarithmic singularity at $x = \pm x_0$.
[6]A. L. Peratt and H. H. Kuehl, Phys. Fluids 15, 1117 (1972).
[7]B. Coppi, Riv. Nuovo Cimento 1, 357 (1969); B. B. Kadomtsev and O. P. Pogutse, Nucl. Fusion 11, 67 (1971).
[8]B. Coppi and A. Taroni, Plasma Phys. 16, 161 (1974).
[9]D. J. Sigmar, private communication.

Resonance Broadening for Wave-Particle and Wave-Wave Turbulence*

N. J. Fisch and A. Bers
Research Laboratory of Electronics and Department of Electrical Engineering and Computer Science, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139
(Received 2 June 1975)

We propose an approximate but simple and general procedure for treating resonance broadening in weak-turbulence interactions. The resonance-broadening corrections to the wave-particle, wave-wave, nonlinear-Landau-damping, and four-wave weak-turbulence equations are computed as examples. The procedure may be used to predict at the outset whether resonance broadening occurs in a specific higher-order process.

The major modification to the conventional weak-turbulence equations demanded by renormalized plasma turbulence theory is a broadening of the resonances.[1-4] A particularly fruitful and powerful approach is to postulate this result and then determine from simple physical considerations the extent of the broadening. In other words, it is decided *a priori* that it is unphysical and incorrect to allow the evolution of the plasma to be determined by quantities describing a "granulation" of wave or particle coordinate space that is finer than a certain resonance-broadening width. It then follows that the weak-turbulence equations are improved through the use of smoothed-over driving quantities

$$\bar{f}_0(v) \equiv \frac{1}{2\delta v}\int_{v-\delta v}^{v+\delta v} f_0(v')\,dv', \qquad \bar{N}(k) \equiv \frac{1}{2\delta k}\int_{k-\delta k}^{k+\delta k} N(k')\,dk', \tag{1}$$

where δv and δk are resonance-broadening widths and reflect the level of turbulence present. (In this Letter we work, for simplicity, in one dimension only.) As an illustrative example, the modified nonlinear-Landau-damping equations could then be put in the form

$$\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial v}\left(D\frac{\partial \bar{f}_0}{\partial v}\right), \quad \frac{\partial N_i}{\partial t} = \gamma_i \bar{N}_i, \tag{2}$$

$$D = \frac{1}{nm}\iint \bar{N}_1\bar{N}_2 H\delta[\omega_1 - \omega_2 - (k_1 - k_2)v]\,dk_1\,dk_2, \tag{3}$$

$$\gamma_i = \int \frac{dk_j \bar{N}_j H}{k_i - k_j}\frac{\partial \bar{f}_0}{\partial v}\bigg|_{v=(\omega_i-\omega_j)/(k_i-k_j)}, \tag{4}$$

where H is a coupling coefficient. Other weak-turbulence equations are similarly modified—obviously, in all cases the modified equations conserve the same quantities as the unmodified equations such as energy, momentum, and particles.[5]

It remains, of course, to calculate the resonance widths for particular interactions. It is expected