1965), pp. 202–207.

[5]We are aware that statistical quark-counting models have been around for a long time. Recent examples are S. Matsuda, Kyoto University Report No. KUNS 315, 1975 (to be published); J. D. Bjorken and G. R. Farrar, Phys. Rev. D 9, 1449 (1974). However, we believe that the dynamical assumptions underlying our model are quite different from those of previous models.

On-Shell Asymptotics of Non-Abelian Gauge Theories*

John M. Cornwall and George Tiktopoulos
Department of Physics, University of California, Los Angeles, California 90024
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In the leading-logarithm approximation, the high-energy fixed-angle elastic fermion-fermion amplitude and the asymptotic form factor have been calculated to sixth order for a Yang-Mills theory; some other processes have been calculated to fourth order. The results suggest properties of exponentiation and factorization which can be described in terms of momentum-dependent anomalous dimensions in the renormalization-group equations. We discuss the possible relation of the zero-mass limit of our results to particle confinement.

We report here calculations based on perturbation theory which suggest certain remarkable asymptotic features of S-matrix elements and form factors for Yang-Mills theories of vectons (vector particles) and fermions. In order that the S matrix exist, of course, it is necessary that the vectons have a mass; this we do by putting in a mass "by hand" in the Feynman gauge.[1] We define the *high-energy fixed-angle regime:* All particles are on shell, and all Lorentz squares of nontrivial momentum sums are large compared to masses, while the ratios of invariants are of order 1. In this regime there is only one large dimensionless quantity, so that our results can be interpreted in terms of the Callan-Symanzik equation (with exceptional momenta).

In the fixed-angle regime the following processes have been calculated in the approximation of retaining only the leading logarithms in each order: (a) fermion-fermion elastic scattering T_{FF} (to sixth order); (b) fermion-vecton elastic scattering T_{FV} (to fourth order); (c) vecton-vecton elastic scattering T_{VV} (to fourth order); (d) the form factor F_F for fermion-antifermion → group-singlet current[2] (to sixth order)[3]; (e) the form factor F_V for vecton-vecton → group-singlet current[2] (to fourth order). The results can be written in the following factored, exponentiated form, valid to the order given above:

$$T_{ij} \sim T_{ij}{}^{B} F_i{}^{n_i/2} F_j{}^{n_j/2}, \tag{1a}$$

$$F_i \sim \exp[(-g^2/16\pi^2)C_i \ln^2(-t/m^2)], \tag{1b}$$

where $i, j = F$ or V, and n_i is the number of legs of type i in T_{ij}. The superscript B denotes the Born approximation, t is the square of the momentum transfer, and we do not distinguish between s, $-t$, and $-u$. The mass scale m is arbitrary, except that if the fermion mass vanishes, m is the vecton mass; C_F and C_V are the eigenvalues of the quadratic Casimir operator for the fermion and vecton (adjoint) representations, respectively.

Details of the calculation will be published elsewhere. We note that the leading logarithms [occurring as powers of $g^2\ln^2(-t/m^2)$] come only from infrared singular regions of integration in momentum space; ultraviolet logarithms [typically powers of $g^2\ln(-t/m^2)$] are nonleading. The contribution from Higgs scalars, ghosts, ordinary scalars, closed fermion loops, and four-vecton vertices (seagull terms) are nonleading, but the three-vecton Yang-Mills coupling is quite important. There are miraculous conspiracies between graphs of very different topology which lead to the factored, exponentiated forms in (1a) and (1b).

For the scattering of *group-singlet* particles or currents (i.e., those which do not couple directly to vectons), we find the striking result that, in a general renormalizable field theory, there are no infrared-singular logarithms; in fact, such amplitudes obey the same renormalization-group equations in the fixed-angle regime that they do in the Euclidean region.[4] Equivalently, the form factors for singlets are identically 1, to *leading* logarithmic order $[g^2\ln^2(-t/m^2)]$. An example which can be studied to all orders is massive quantum electrodynamics; the leading logarithms are as shown in (1a) and (1b), with $C_F=1$ and $C_V=0$.[5] It is essential to note that this holds true for "composite" group-singlet states composed of two or more elementary nonsinglet particles; here by composite we mean that there is no large momentum transfer between any pair of constituents.

The exponentiated, factored forms (1a) and (1b) suggest an interpretation in terms of effective renormalization-group equations, in which each on-shell particle is assigned a *momentum-dependent anomalous dimension.* These dimensions add coherently for composite states (i.e., one sums the group charges of the constituents before squaring) unlike ordinary anomalous dimensions which add incoherently. As an exercise to which we do not necessarily attach any physical significance we show how the Callan-Symanzik (CS) equation is solved in the presence of momentum-dependent anomalous dimensions. We use the CS equations for a proper on-shell Green's function G^{n_i} with n_i legs of type i ($i=F$, V, singlet). Because the momenta are exceptional, the mass-insertion term ΔG^{n_i} must be saved; indeed it is the most singular term for a vector theory. The CS equation is, in conventional notation,

$$\left(\sum m\frac{\partial}{\partial m}+\beta\frac{\partial}{\partial g}-\sum n_i\gamma_i\right)G^{n_i}=\Delta G^{n_i}. \tag{2}$$

Our calculations so far do not contain effects due to ultraviolet singularities, so that the leading mass insertion term comes from a comparison of (1) and (2), setting $\beta=\gamma_i\equiv 0$. Then the CS equation may be rewritten as an effective renormalization group equation,

$$\left(\sum m\frac{\partial}{\partial m}+\beta\frac{\partial}{\partial g}-\sum n_i\Gamma_i\right)G^{n_i}=0, \tag{3}$$

in which each on-shell particle (or singlet current, on-shell or off-shell) is assigned a momentum-dependent anomalous dimension:

$$\Gamma_i=(g^2/8\pi^2)C_i\ln(-t/m^2)+\dots, \tag{4}$$

where the omitted terms, which include the γ_i as well as other terms which may depend on ratios such as t/s, are nonleading.[6] Group-singlet particles, whether on-shell or off-shell, have $C_i\equiv 0$ in (4). It is not difficult to show that the general solution to (3) is

$$G(z)=R(\bar{g}(z))\exp[-\int_0^z dz'\sum n_i\Gamma_i(\bar{g}(z');z-z')], \tag{5}$$

where $z=\frac{1}{2}\ln(-t/m^2)$, $\bar{g}(z)$ is the effective charge, normalized so that $\bar{g}(0)=g$, and R is arbitrary. For asymptotically free[7] theories $\bar{g}(z)\sim 0$, so that R can be chosen as the Born approximation to G. With use of (4) in (5), the leading terms in the form factors for asymptotically free theories ($\beta=-bg^3$, $b>0$) turn out to be[8]

$$F_i(-t/m^2)\sim\exp[-(C_i/8\pi^2b)\ln(-t/m^2)\ln\ln(-t/m^2)]. \tag{6}$$

Whatever the corrections to the results (1a) and (1b) are, they are irrelevant to the scattering of group-singlet particles. Here the quark-counting rule[9] holds, aside from short-distance corrections of the sort that violate Bjorken scaling,[10] if the composite wave functions yield no singularities.[9] It may well be true that fixed-angle hadronic amplitudes are products of electromagnetic form factors times a function of t/s.[11]

Why does exponentiation and factorization occur? We have been able to derive (1a) and (1b) by assuming that the Yang-Mills currents obey naive Ward identities *at large energies* (recall that ghost-line graphs are nonleading) and by deriving analogs of the Low theorem valid for massless internal vecton lines. This line of argument can be followed in great detail in the Abelian case; we shall publish it in detail elsewhere. It can also be shown in the Abelian case that (1a) and (1b) follow from the assumption that only two-particle unitarity cuts are dominant in the asymptotic form factor and fixed-angle amplitudes.

Exponentiation would follow if we could relate the infrared anomalous dimensions to conventional anomalous dimensions. It is known that, in asymptotically free theories, the moments of the electroproduction structure function $F_2(x,q^2)$ behave, for large $-q^2$, as

$$\int_0^1 dx\,x^N F_2(x,q^2)\sim C_N[\ln(-q^2/m^2)]^{-a_N}, \tag{7}$$

where $a_N=\gamma_N/2g^2b$, and the γ_N are anomalous dimensions of operators occurring in the Wilson expansion.[10] For large N, these dimensions are

$$\gamma_N\sim(g^2/2\pi^2)C_F\ln N \tag{8}$$

and this holds for singlet and nonsinglet operators, with the operator-mixing problem[10] disappearing for large N. In perturbation theory, $C_N \sim 1$ for large N.

Clearly, for large N only those terms in F_2 survive in (7) for which the support is concentrated near $x=1$; these are the one-fermion-reducible graphs, with slightly off-shell form factors. By using a casual spectral representation for the off-shell form factors, we have been able to show that if N is replaced by $-q^2/m^2$ in (7), the moments of F_2 asymptotically approach $F_F^2(-q^2/m^2)$. Using (8), a simple calculation shows that the result (6) is obtained for the fermion form factor. Incidentally, this approach reveals the absence of anomalous infrared effects in field theories without gauge vector particles, where it is known[12] that the anomalous dimensions in the Wilson expansion vanish for large N.

On the basis of our results it is natural to speculate that the zero-mass limit of Yang-Mills theories leads to particle confinement. Indeed, for fixed external momenta, as both the vecton mass m_V and the fermion mass m_F approach zero all exclusive amplitudes vanish. Moreover, since the vanishing of multiparticle amplitudes becomes faster with increasing number of produced particles (asymptotic additivity of momentum-dependent anomalous dimension), the *inclusive* cross sections should also vanish, in contrast to the Abelian case where m_V is replaced by an experimental energy resolution.

A less drastic limit is the usual infrared one: $m_F \neq 0$, external momenta fixed, $m_V \to 0$. In this case leading-logarithm calculations (powers of $g^2 \ln^2 m_V$) indicate that the vectons are still confined but not necessarily the fermions. The nonleading infrared powers of $g^2 \ln m_V$ must be examined in order to decide the fate of the fermions but if the ordinary quantum electrodynamic infrared problem is any guide we may still expect confinement of fermions. The reason is that the soft real quanta which cancel part of the infrared singularity in quantum electrodynamics are not available in the non-Abelian case since they are confined. The infrared singularity of the virtual quanta is thus laid bare and results in a vanishing of cross sections for fermions.

Of course, all this is based on the hope that leading-logarithm effects appear multiplicatively rather than additively in the complete perturbation series, i.e., that nonleading effects appear as additive corrections to Γ_t in (2).

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[1]Besides the usual Higgs-Kibble mechanism for mass generation, there are other ways of making the S matrix finite which are compatible with asymptotic freedom, which we consider later on. One is spontaneous breakdown without scalars. Another is confinement, which may be inherent to those Yang-Mills theories such as the usual color theories that cannot undergo spontaneous breakdown [J. M. Cornwall, Phys. Rev. D 10, 500 (1974)]: The confined vector mesons (and possibly fermions) are slightly off the zero-mass shell, which is essentially equivalent to giving them a mass.

[2]An example is the fermion electromagnetic form factor in a Yang-Mills theory of colored quarks, where the electromagnetic current is a color singlet. Analogous objects can be constructed for the Yang-Mills vectons. The spin of the group-singlet current is irrelevant.

[3]After the original submission of this work, an article appeared [J. J. Carrazone, E. R. Poggio, and H. R. Quinn, Phys. Rev. D 11, 2286 (1975)] claiming that exponentiation for the form factor F_F fails in sixth order. We have calculated all the sixth-order graphs by standard Feynman-parameter techniques and have found two graphs [their Fig. 12(a) and the unique graph with three three-vecton vertices] to have been given erroneous values in the Sudakov region $(-t \gg p^2, p'^2 \gg M^2)$. The graph of Fig. 12(a) should be divided by 3, and the sign of the other graph changed. The result is that exponentiation works through sixth order both in the Sudakov region and in the mass-shell region considered here. Professor Quinn has informed us that she now agrees that the form factor exponentiates through sixth order. We have heard from S. J. Brodsky that similar results have been gotten by Lipatov *et al.* (unpublished).

[4]G. Tiktopoulos, Phys. Rev. D 11, 2252 (1975). The absence of infrared logarithms for form factors of singlets has been noted by G. C. Marques, Phys. Rev. D 9, 386 (1974), for $\bar{\psi}\psi\varphi$ theories. It was conjectured long ago by K. Huang and F. E. Low, Zh. Eksp. Teor. Fiz. 46, 845 (1963) [Sov. Phys. JETP 19, 579 (1964)] that this would be the case for fixed-angle scattering in nonvector theories.

[5]Results equivalent to (1a) for T_{FF} and (1b) for F_F in the Abelian case were suggested by H. M. Fried and T. K. Gaisser, Phys. Rev. 179, 1491 (1969), and J. L. Cardy, Nucl. Phys. B33, 139 (1971), although we are not in complete agreement with the treatments of these authors. A recent preprint by I. G. Halliday, J. Huskins, and C. T. Sachrajda is in full agreement with our calculations for the Abelian case, including the absence of the notorious Halliday-Landshoff pinch singularity.

[6]In ordering the terms in a leading-logarithm expansion, or in the renormalization group, it is appropriate to think of $g^2/4\pi$ as being of the same order as $[\ln(-t/m^2)]^{-1} \sim \epsilon \ll 1$. Thus the leading term in (2) is of order 1, and nonleading terms of order ϵ or higher.

[7]D. Gross and F. Wilczek, Phys. Rev. Lett. 30, 1343 (1973); H. D. Politzer, Phys. Rev. Lett. 30, 1346 (1973).

[8]The same result has been found by A. De Rújula, Phys. Rev. Lett. 32, 1143 (1974), and by D. Gross and

S. B. Treiman, Phys. Rev. Lett. 32, 1145 (1974), by consideration of the large-N limit of the anomalous dimensions of operators in the Wilson expansion of two electromagnetic currents. These authors work with the matrix elements of composite objects, which have a power-law decrease $(q^2)^{-a}$ multiplied into (6). This is to be omitted for elementary fermions. We emphasize that in our approach *composite* color-singlet states have *no* anomalous momentum-dependent dimensions.

[9]S. J. Brodsky and G. Farrar, Phys. Rev. Lett. 31, 1153 (1973), and Phys. Rev. D 11, 1309 (1975); V. A. Matveev, R. M. Muradyan, and A. N. Tavkhelidze, Lett. Nuovo Cimento 5, 907 (1972), and 7, 719 (1973).

[10]D. Gross and F. Wilczek, Phys. Rev. D 9, 980 (1974); H. Georgi and H. D. Politzer, Phys. Rev. D 9, 416 (1974).

[11]Such a form for wide-angle scattering was suggested, in the context of πN field theory with the approximation $F_\pi(t) \equiv 1$, by J. M. Cornwall and D. J. Levy, Phys. Rev. D 3, 712 (1971). More recently, a number of authors have discussed fixed-angle scattering in theories with no vector particles; we mention in addition to those of Ref. 4, M. Creutz and L.-L. Wang, BNL Report No. BNL-19078, 1974 (unpublished); S. Shei, to be published; C. G. Callan and D. J. Gross, to be published; J. M. Corenstein, to be published. There is a large amount of work on parton models which can be traced from the work of R. Blankenbecler and S. J. Brodsky, Phys. Rev. D 10, 2973 (1974), and P. V. Landshoff and J. C. Polkinghorne, Phys. Rev. D 8, 4157 (1973).

[12]N. Christ, B. Hasslacher, and A. Mueller, Phys. Rev. D 6, 3543 (1972).

$\alpha^2(Z\alpha)^2$ Vacuum-Polarization Correction in Muonic Atoms*

Dwight H. Fujimoto
Physics Department, University of Washington, Seattle, Washington 98195
(Received 12 May 1975)

The order-$\alpha^2(Z\alpha)^2$ vacuum-polarization correction for transition energies in muonic atoms is calculated in the static-muon, massless-electron limit. Conventional Feynman-parametric-integral techniques are utilized. The correction is found to be negligibly small, contrary to a previous claim appearing in the literature.

The discrepancy between observational results and quantum electrodynamic theory (QED) in muonic systems has been thoroughly discussed.[1] For example, for the $5g$-$4f$ transitions of muonic Pb, the experiments reviewed in Ref. 1 show roughly a 2-standard-deviation discrepancy of about 1 part in 10^4 where the observed x-ray energies are approximately 50 eV less than the QED predictions. The apparent failure of QED in a strong Coulomb field has led to new theoretical investigations, among which there are two with strikingly contradictory results. Chen[2] has claimed that the energy discrepancy for muonic atoms of heavy nuclei is essentially eliminated after the traditionally neglected two-photon vacuum-polarization corrections of order α^2 are included. In particular, for the Pb example given above, Chen has calculated a correction of -35 eV due to the $\alpha^2(Z\alpha)^2$ vacuum-polarization process illustrated by Fig. 1. On the other hand, Wilets and Rinker[3] have done calculations from which they are able to estimate that this same correction is less than 3 eV in magnitude and of the opposite sign. In order to help resolve the theoretical disagreement on the size of the $\alpha^2(Z\alpha)^2$ vacuum-polarization effect in muonic atoms, I present a third calculation in this Letter. The method used here is quite different from the approaches used in the previous investigations, and the answer is found to be in complete agreement with the estimate made by Wilets and Rinker.

The present investigation into the process of Fig. 1 utilizes conventional Feynman-parametric-integral techniques[4] together with a judicious set of approximations[5] which reduces the calculation-

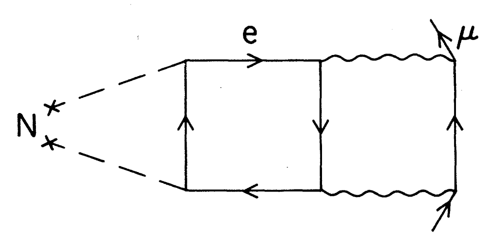

FIG. 1. The two-photon vacuum-polarization correction to order $\alpha^2(Z\alpha)^2$ for a muonic atom (μ-N). The dashed lines represent the Coulomb interaction with a static point charge, and the wavy lines are photons. Permutations of the light-by-light scattering subgraph must be included also.