Renormalization-Group Approach to Percolation Problems*

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The relation between the s-state Ashkin-Teller-Potts (ATP) model and the percolation problem given by Fortuin and Kasteleyn is used to formulate a renormalization-group treatment of the percolation problem. Both an ϵ expansion near 6 spatial dimensions and cluster approximations for the recursion relations of a triangular lattice are used. Series results for the ATP model are adapted to the percolation problem.

In the last few years the renormalization group[1,2] (RG) has provided a detailed understanding of phase transitions in a large number of model systems. In this Letter we will show how this approach can be applied to the percolation problem.[3] The systems we study are described as follows: For the site (bond) problem we consider a lattice in which sites (bonds) are randomly occupied with probability p and are vacant with probability $q = 1 - p$. For $p < p_c$ occupied sites (bonds) form unconnected clusters. The mean square cluster size $S(p)$ diverges as $p \to p_c$ and for $p > p_c$ a fraction of sites (bonds), $P(p)$, are in an infinitely large cluster. This "phase transition" at $p = p_c$ differs from the usual ones treated by the RG in that (a) the system is not described by a Hamiltonian, and (b) there is no obvious analog of the partition function.

The basis of our work is a series of papers by Kasteleyn and Fortuin[4] who consider an "s-state" Ashkin-Teller-Potts (ATP) model defined by the Hamiltonian

$$\mathcal{H}^{(s)} = -J \sum_{\langle ij \rangle} (s\delta_{n_i n_j} - 1) - h \sum_i (s\delta_{n_i 1} - 1), \quad (1)$$

where for each site i the variable n_i assumes the values $1, 2, \ldots, s$, the sum $\langle ij \rangle$ is over nearest-neighbor pairs, and δ_{nm} is the Kronecker delta. For $s = 2$ Eq. (1) is an Ising Hamiltonian. Alternatively $\mathcal{H}^{(s)}$ can be written in terms of vectors $\vec{v}_i$ confined to point in the direction from the center of a multihedron towards one of its vertices. In this realization

$$\mathcal{H}^{(s)} = -J(s-1) \sum_{\langle ij \rangle} \vec{v}_i \cdot \vec{v}_j - (s-1) \sum_i \vec{v}_i \cdot \vec{h}, \quad (2)$$

since $\vec{v}_i \cdot \vec{v}_j = 1$ if $\vec{v}_i \| \vec{v}_j$, and $\vec{v}_i \cdot \vec{v}_j = -(s-1)^{-1}$ otherwise. The ATP model and the percolation problem are related by

$$P(p) = (s-1)^{-1} \partial F / \partial h |_{h=s-1=0}, \quad (3a)$$

$$S(p) = (s-1)^{-1} \partial^2 F / \partial h^2 |_{h=s-1=0}, \quad (3b)$$

where $q = e^{-\beta J}$ and F is the free energy per site for the s-state model. These relations have been used only in a qualitative way up to now: i.e., to state that[5] $P(p) \sim |p - p_c|^{\beta}$ and $S(p) \sim |p - p_c|^{-\gamma}$ as $p \to p_c$. The usefulness of the relation between percolation and the ATP model is that all the apparatus, e.g. series expansions, RG, etc., used for a Hamiltonian formulation can be applied to the percolation problem. This program has not been proposed until now.

A simple test and application of these ideas is to use Eq. (3) to generate power-series expansions in q for P, S, and "specific heat" C from the low-temperature series expansion given for

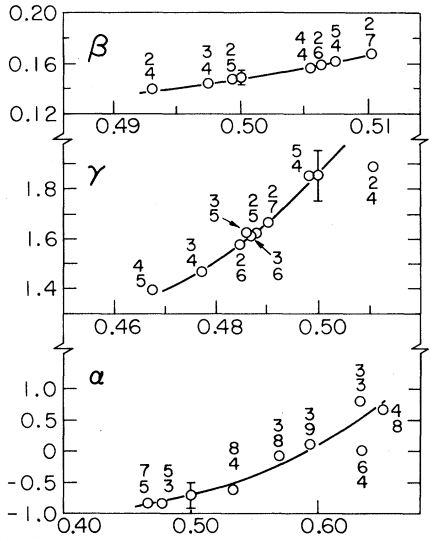


FIG. 1. Results of Padé approximants for bond dilution on the square lattice. The exponents β, γ, and α as obtained from the residues of the Padé approximants $P_N(q)/Q_M(q)$ are plotted versus the location of the zero of $Q_M(q)$ near $q_c = 0.5$. The values of N and M for each point are given (N is the upper number, M the lower). The solid line is a smooth curve through the points. The value of the exponent with an associated uncertainty is given by the bar obtained from the solid curve at the exact value of $q_c = 0.5$.

the simple square lattice by Straley and Fisher[6] and Kihara, Midzuno, and Shizume.[7] We then formed Padé approximants to the functions expected to have simple poles, i.e.

$$\frac{d}{dq}\ln\left(\frac{dC(q)}{dq}\right), \quad \frac{d\ln P(q)}{dq}, \quad \text{and} \quad \frac{d\ln[q^{-4}S(q)]}{dq},$$

as done in Ref. 6 for $s=3$. The residues at the poles of the respective Padé approximants at $q=q^*$ give the corresponding exponents α, β, and γ. Since the values of the exponents so obtained are correlated with the corresponding values of q^* (see Fig. 1) we determined the residues from their values at the exact[8] value of q^*: viz. $q^*=0.5$. The results are summarized and compared to previous results[9-11] in Table I.

TABLE I. Numerical results for exponents for the square lattice.

	Series in q for bond dilution	Monte-Carlo simulation for site dilution		Series in p for site dilution [b]
		This work	Dean-Bird [a]	
α	-0.7 ± 0.2			
β	0.148 ± 0.004	0.16 ± 0.02 [c]	0.14 ± 0.02	
γ	1.58 ± 0.2	2.1 ± 0.2	1.9 ± 0.2	2.1 ± 0.2
ν		1.5 ± 0.2		

[a] These exponents were obtained by analyzing the data of Ref. 9. This result for β agrees with that found by S. Kirkpatrick (unpublished).

[b] See Ref. 10.

[c] This value is not inconsistent with the results of Ref. 11.

We have also used the RG to treat the s-state model. One technique we used was the cluster approximation for the recursion relations due to Niemeyer and van Leeuwen.[12] In that scheme it is convenient to consider triangular cells on a triangular lattice. One renormalization step consists in summing over all variables in a cell subject to the cell variable n_i under the assumption of a fixed value Q. Thus for each Q there are s^2 configurations as depicted in Fig. 2. For a two-cell cluster A-B the recursion relation defined by invariance of the partition function is, for $s=1$,

$$x_{AB}' = 2x^2 + 2x^3 - 5x^4 + 4x^6 - 2x^7, \tag{4a}$$

where $x' = e^{-J'/kT}$ is the renormalized cell-cell interaction and $x = e^{-J/kT}$ is the initial site-site interaction. For the two-cell cluster A-C the recursion relation is

$$x_{AC}' = 2x^2 + x^3 - 4x^4 + 5x^6 - 4x^7 + x^8. \tag{4b}$$

The appropriate averaged renormalized interaction is then $x' = (x_{AB}')^{1/3}(x_{AC}')^{2/3}$. A similar treatment of three-cell clusters following the method of Niemeyer and van Leeuwen and relating cell and site variables as in the two-cell–cluster approximation gives the results shown in Table II. In this approximation we had to allow for a renormalized three-site interaction of the form

$$V^{(3)} = L[s^2\delta_{n_i n_j}\delta_{n_j n_k} - s(\delta_{n_i n_j} + \delta_{n_j n_k} + \delta_{n_i n_k}) + 2] \tag{5}$$

for sites i, j, and k being mutual nearest neighbors. At present our results are very crude as indicated by the rather poor value of $x^* \equiv q_c$ we get, 0.792, compared to the exact result,[8] $q_c \cong 0.653$. Thus the Niemeyer–van Leeuwen scheme seems to work slightly less well here than for the Ising model.[12] The current results for ν, which is quite sensitive to the size of the thermal eigenvalue, are not at all reliable.

Before proceeding to the ϵ expansion, let us investigate the mean-field theory for the s-state ATP model. In the ground state of Eq. (2), all $\vec{v}_i$ are parallel and lie along one of the multihedral directions $\hat{n}_j$. Thus, in the ordered state, there will be an order parameter $\vec{v} = v\hat{n}_i$. For $s \neq 2$ $v<0$ corresponds to $\vec{v}$ lying along a direction which is not equivalent to a multihedral direction $\hat{n}_j$. Accordingly, for $s \neq 2$ we still only consider mean-field solutions having $v>0$. Within the RG this restriction is related to the fact that the recursion relations do not connect the region in these spaces for which w in Eq. (7) below is positive with that for which $w<0$. The mean-field free energy per site for Eq. (2) to fourth order in v is[13]

$$(s-1)^{-1}F = \tfrac{1}{2}(T - T_c)v^2 - \tfrac{1}{6}T(s-2)v^3 + \tfrac{1}{12}T(s^2 - 3s + 3)v^4, \tag{6}$$

where T is the temperature and T_c is the mean-field transition temperature. At $s=2$, the cubic term disappears and the usual mean-field theory for the Ising model is regained. For $s>2$, the cubic term is negative for positive v, and a first-order transition takes place.[13] For $s<2$, the cubic term is posi-

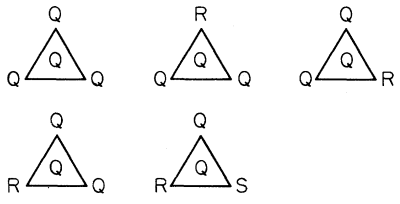
tive for positive v and a continuous transition occurs with $v = +2(T_c - T)/|s-2|T$ for $T < T_c$, i.e. $\beta = 1$. Also one finds $\gamma = 1$ and $\nu = \frac{1}{2}$. The exact solution of the percolation problem on the Cayley tree yields the same values for these exponents.[14,15] Using these results, Toulouse[15] predicted that if the exponents for the percolation problem are to become nonclassical, they should do so at six dimensions. He argued that the Cayley tree is an infinite dimensional lattice, and that the exponents ν, β, and γ should maintain their infinite-dimensional values down to a critical dimension d_c at which the scaling relation $d\nu = 2\beta + \gamma$ should be satisfied. From this, one finds $d_c = 6$. We expect, therefore, that nonclassical exponents for the percolation problem can be obtained by expanding in $\epsilon = 6 - d$.

In order to apply the ϵ expansion, we need a continuum generalization of the s-state ATP model. Such a generalization was introduced by Priest and Lubensky[16] following Golner[17] and involves *diagonal* traceless tensors Q_{ij} of dimension s. The Hamiltonian for this model is

$$\mathcal{H} = \tfrac{1}{2}\int d^dx\,(r\,\mathrm{Tr}Q^2 + \sum_{ijk}\nabla_i Q_{jk}\nabla_i Q_{jk}) + \int d^dx\,[w\,\mathrm{Tr}Q^3 + u(\mathrm{Tr}Q^2)^2 + v\,\mathrm{Tr}Q^4]\,. \tag{7}$$

If $w = 0$, this Hamiltonian has a fixed-point structure in $4-\epsilon$ dimensions that is very similar to that of the hypercubic model.[18] For $w \neq 0$, there is a stable fixed point in $6-\epsilon$ dimensions with $u^* = v^* = O(\epsilon^2)$ and $(w^*)^2 = \epsilon s/18K_d(10-3s)$, where $K_d^{-1} = 2^{d-1}\pi^{d/2}\Gamma(d/2)$, and exponents $\eta = -\frac{1}{21}\epsilon$, $\nu = \frac{1}{2} + \frac{5}{84}\epsilon$, $\gamma = 1 + \frac{1}{7}\epsilon$, and $\beta = 1 - \frac{1}{7}\epsilon$ for $s = 1$. We feel that these exponents can safely be identified with the percolation exponents though the stability of the Hamiltonian at the fixed point for large $\mathrm{Tr}Q^2$ has not yet been thoroughly investigated.

It is tempting to identify the above fixed point at $s = 3$ with the continuous transition that is known to occur in the ATP models in two dimensions.[19] We hesitate to make this identification because of the absence of a mean-field theory predicting a second-order transition in that case.



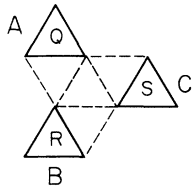


FIG. 2. Top: Possible values of site variables subject to the cell variable being Q. Here Q, R, and S are all distinct. Bottom: Illustration of cells used for the two-cell and three-cell cluster approximation for the RG.

Finally, we have performed a series of simulations of the percolation problem by generating pseudo random configurations of randomly occupied sites on a square lattice. In this way we could estimate $P(p)$ and $S(p)$ for $p \to p_c$. In addition we could evaluate the average, denoted by $\langle\ \rangle$, over configurations and over R of the probability, $g(R, R+r)$, that R and $R+r$ are in the same cluster of occupied sites. Our results are consistent with the form $\langle g(R, R+r)\rangle \sim r^{2-d-\eta} \times \exp[-r/\xi(p)]$ but by no means prove its correctness. Furthermore, we plotted ξ versus $|p-p_c|$ to get ν. Because of the finite sample size used in the studies it was not possible to determine η and ν with great accuracy. It was possible however to determine $\nu(\eta)$: That is, given the value of η, ν could be determined. The values in Table I correspond to the intersection of such a curve with the curve $\eta\nu = 2\beta$ (scaling law). Alternatively, for each configuration we plotted $\ln S$ versus $\ln\xi$ to get η since we expect that $S(p) \sim \xi(p)^{2-\eta}$. Indeed we found $\ln S \propto \ln\xi$ did hold configuration by configuration in spite of the occurrences of fluctuations in the average concentration about its nominal value but the value of η we found from this procedure ($\eta \approx 0.5$) is not reliable.

It appears that $\beta = 0.15$ is the most accurately

TABLE II. Results of cluster approximations to the RG for triangular lattice.

No. of cells	Fixed-point values $e^{-J/kT}$	$e^{-L/kT}$	ν	q_c
2	0.664		2.4	0.664
3	0.786	1.014	3.0	0.792 [a]

[a] Calculated as the value of q on the critical surface for which $L = 0$.

known exponent.[20] If one sets $\alpha = -0.6$, $\nu = 1.3$, $\gamma = 2.3$, and $\eta = 0.23$, all of which values are within the error ranges presently obtained, then the various scaling relations, $\alpha = 2 - d\nu$ and $\alpha + 2\beta + \gamma = 2$, are satisfied. Hence these values represent the most reasonable estimates we can make at present. It is interesting that η appears to depend only weakly (if at all) on s.

The numerical results of the various approaches are not yet consistent, probably because the RG methods have not yet been sufficiently refined. However, other finite-cluster RG schemes[21] are being investigated and will no doubt lead to improved accuracy.

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Search for Gravitational Radiation at 145 Hz*

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We present the results of gravitational radiation experiments with two 1400-kg antennas at $\nu_0 = 145$ Hz. The square antennas, $1.65 \times 1.65 \times 0.19$ m^3, $Q_M \simeq 2.0 \times 10^5$, with electrostatic transducers of $\beta \simeq 1.8 \times 10^{-3}$, have mean effective noise energy $\bar{D} \simeq 0.034 kT_r$. The observed cross correlation of the outputs of the two detectors gives an upper limit for the relation between the mean energy spectrum density $F(\nu_0)$ and the daily occurrence rate $\dot{N}_G$ of gravitational radiation pulses: $F(\nu_0)\dot{N}_G^{1/2} \leq (3.6^{+1.0}_{-1.4}) \times 10^6$ J m^{-2} Hz^{-1} day$^{-1/2}$.

Large energy fluxes of gravitational radiation (GR) observed by Weber[1] at 1660 Hz have not yet been confirmed.[2] Search for GR in other frequency regions should help settle the issue and determine the GR spectrum distribution, if any. We report here the results of the correlation measurement on GR performed with two 1400-kg antennas[3] at 145 Hz.

The square antennas, $1.65 \times 1.65 \times 0.19$ m^3 with a cut on each side (Fig. 1), are fabricated from aluminum alloy (52SR) plates. The structure has symmetry D_{4h}. The fundamental in-plane vibra-