An integration of (9) gives

$$\Phi'' + \Phi'^2 - \Phi'\Lambda' + 2r^{-2}(\Phi' - \Lambda') + r^{-2}(1 - e^{2\Lambda}) + \tfrac{1}{2}R_0 e^{2\Lambda} = 0, \tag{10}$$

where R_0 is the constant scalar curvature. Both (8) and (10) are second order in Λ and Φ'; hence the general solution of (8) and (10) has four arbitrary parameters. (Because of nonlinearity there might be a discrete number of such sets of four parameters.) The additive constant in Φ can be removed by a change of time scale. Therefore the general static spherical-symmetric solution has four arbitrary parameters. This demonstrates that the solution of (1) is much richer than that of (2) (two parameters) or (3) (one parameter). As a matter of fact

$$ds^2 = -dt^2 + (1 + c_1/r + c_2 r^2)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) \tag{11}$$

is a solution of (1) but not of (2) or (3); (11) possesses no gravitational red shifts.[7] The problems of boundary conditions and sources for (1) deserve extensive studies to clear up this richness of solutions. In view of the present success of the renormalization of gauge theories, these studies could contribute to the solution of the long-standing problems of quantum gravity.

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[5]The second term in (2) is a constant term and, hence, does not contribute to (4).

[6]In view of this result, the recent claim of Thompson [Phys. Rev. Lett. 34, 507 (1975)] that the Birkhoff theorem of general relativity generalizes to (1) is invalid.

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Geometrically Degenerate Solutions of the Kilmister-Yang Equations

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I solve, in principle, the Kilmister-Yang equations for the degenerate cases of conformal flatness and decomposability of space-time. The unphysical metrics discussed by Pavelle belong to these degenerate classes. Perhaps using the methods outlined here it will be possible to determine if these "unphysical" metrics are isolated examples or if they are typical of such "geometric-degenerate" classes.

In a recent paper Pavelle[1] has discussed certain solutions of the Kilmister-Yang (KY) equations[2,3] and has argued that in particular conformally flat solutions to these equations should not be allowable in the theory. Of course conformally flat solutions of the Einstein vacuum equations are necessarily flat and hence no additional constraint of this type is necessary there. Furthermore another unphysical solution discussed by Pavelle is also degenerate in the sense that it possesses a timelike parallel vector field and hence is decomposable. In fact almost all the solutions exhibited recently[1,4,5] possess some degenerate "geometric" property which is ruled out by the field equations in the orthodox theory (i.e., the only solutions with such properties are necessarily flat).

I present here some general theorems on conformally flat and decomposable spaces which should enable the construction of many solutions to the KY equations and hence a fuller discussion of the unphysical nature of the types of solution discussed by Pavelle. We consider the local differential geometry of an n-dimensional Riemannian space V_n with metric g_{ab}. (For a V_4 we assume of course the Lorentz signature but the results of the next two sections are true in general.) By a KY space we will mean a V_n which satisfies (locally)

$$R_{bcd;a}{}^{a}=2R_{b[c;d]}=0. \tag{1}$$

Conformally flat KY spaces.—*Theorem 1*: A V_3 is a KY space if and only if it is conformally flat with constant scalar curvature.

Theorem 2: For $n\geq 4$, every conformally flat V_n with constant scalar curvature is a KY space.

The metric of every V_2 can (locally) be expressed (modulo signature)

$$ds_2{}^2=\epsilon e^{2\sigma}[(dx^1)^2\pm(dx^2)^2],\quad \epsilon=\pm 1, \tag{2}$$

with $\sigma=\sigma(x^1,x^2)$. Hence in two dimensions we have *Theorem 3*: A V_2 is a KY space if and only if it has constant scalar curvature, and consequently is a space of constant curvature.

Decomposable KY spaces.—A space V_m is said to be n by $m-n$ decomposable as the direct product $V_n\times V_{m-n}$ if and only if there exists a coordinate system $x^1,\dots,x^n,x^{n+1},\dots,x^m$ such that the metric of V_m assumes the form

$$ds^2=g_{ab}dx^a dx^b=g_{\alpha\beta}dx^\alpha dx^\beta+g_{AB}dx^A dx^B, \tag{3}$$

where Greek letters take the values $1,2,\dots,n$, and Latin capitals the values $n+1,\dots,m$, and where $g_{\alpha\beta}=g_{\alpha\beta}(x^\alpha)$, $g_{AB}=g_{AB}(x^C)$. If one of the components (e.g., V_{m-n}) in the direct product $V_n\times V_{m-n}$ is flat, then V_m is called a flat extension of a V_n.

The following are easily verifiable:

Theorem 4: Every flat extension of a KY space is a KY space.

Theorem 5: The direct product of two KY spaces is a KY space.

Theorem 6: If $V_r{}^{(1)}$ and $V_n{}^{(2)}$ are Einstein spaces $[R_{ab}\propto g_{ab}]$ with scalar curvatures $R^{(1)}$ and $R^{(2)}$, respectively, then their direct product is an $(n+r)$-dimensional KY space. In particular it is an Einstein space if and only if $R^{(1)}/r=R^{(2)}/n$.

Theorem 7: If a four-dimensional KY space is decomposable then it is one of the following types: (i) It is the direct product of two two-dimensional spaces of constant curvature and is not flat; (ii) it is the flat extension of a conformally flat V_3 with constant scalar curvature and is not flat; or (iii) it is flat. We note that types (i) and (ii) do not exist for the orthodox Einstein vacuum equations.

Particular solutions.—Applying the above theorems we are able to construct several classes of solutions to the field equations (1).

(a) The conformally flat V_3 with metric

$$ds_3{}^2=-p^4(\delta_{ab}dx^a dx^b),\quad a,b=1,2,3$$

(and signature -3), has a scalar curvature

$$R=-8p^{-5}\delta^{ab}p_{,ab}.$$

Consequently the decomposable V_4,

$$ds^2=dt^2-p^4(\delta_{ab}dx^a dx^b), \tag{4}$$

is a KY space iff p satisfies the differential equation

$$\nabla^2 p+\tfrac{1}{8}Rp^5=0, \tag{5}$$

where R is a constant and ∇^2 denotes the three-dimensional Laplacian.

If we assume the usual spherical polar coordinates (r,θ,φ) and $p=p(r)$, Eq. (5) becomes

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dp}{dr}\right)+\frac{R}{8}p^5=0. \tag{6}$$

A singular solution of (6) is given by $p=(2/R)^{1/4}\times r^{-1/2}$ from which results the metric

$$ds^2=dt^2-2R^{-1}r^{-2}(dr^2+r^2d\theta^2+r^2\sin^2\theta\, d\varphi^2).$$

Under the change of variables $p(r)\equiv(\frac{1}{8}R)^{-1/4}\times\exp(\frac{1}{2}q)v(q)$, $q=-\ln r$, Eq. (6) is

$$d^2v/dq^2-\tfrac{1}{4}v+v^5=0, \tag{6a}$$

which has a first integral

$$(dr/dq)^2=\tfrac{1}{4}v^2-\tfrac{1}{3}v^6+C_1.$$

With $C_1\equiv 0$ the above integrates to give

$$v^2=\tfrac{1}{2}\sqrt{3}\,\mathrm{sech}(q+C_2),$$

which in the original variables yields

$$p^2=(\tfrac{1}{8}R)^{-1/2}2k\sqrt{3}(4k^2+r^2)^{-1},\quad 2k\equiv\exp C_2,$$

and results in a line element isometric to

$$ds^2=dt^2-(1+r^2/4k^2)^{-2}(dr^2+r^2d\theta^2+r^2\sin^2\theta\, d\varphi^2).$$

Under the transformation $\rho=r(1+r^2/4k^2)^{-1}$, this becomes

$$ds^2=dt^2-(1-\rho^2/k^2)^{-1}d\rho^2-\rho^2d\theta^2-\rho^2\sin^2\theta\, d\varphi^2,$$

the line element of the Einstein Universe.

For $R\equiv 0$, the solution of (6) is given by $p=\alpha+\beta/r$ which results in a flat metric if either α or β is zero. Otherwise the line element is isometric with

$$ds^2=dt^2-(1+m/r)^4(dr^2+r^2d\theta^2+r^2\sin^2\theta\, d\varphi^2),$$

and under the transformation $\rho=(r+m)^2/r$ is

$$ds^2=dt^2-(1+4m/\rho)^{-1}d\rho^2-\rho^2 d\theta^2-\rho^2\sin^2\theta\, d\varphi^2.$$

This is therefore an unphysical metric [Eq. (4)] discussed by Pavelle.[1]

By similar specialization in Eqs. (4) and (5), static solutions with cylindrical and axial symmetry (for example) can be generated.

(b) The class of conformally flat solutions of the KY equations is determined by solutions of the differential equation

$$\Box^2 p=\tfrac{1}{6}Rp^3. \tag{7}$$

Here R is the *constant* scalar curvature of the V_4 with line element

$$ds^2=p^2[dt^2-(dx^1)^2-(dx^2)^2-(dx^3)^2],$$

and $\Box^2$ is the four-dimensional D'Alembertian operator. Hence in the special case $R\equiv 0$, the conformally flat KY spaces are determined by solutions of the wave equation.

With spatial spherical polar coordinates (r,θ,φ) and the assumption that $p=p(r)$, Eq. (7) becomes

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dp}{dr}\right)+\frac{R}{6}p^3=0, \tag{7a}$$

which for $R\equiv 0$ has the general solution

$$p=\alpha+\beta/r \tag{8}$$

(α and β constants). For $\alpha\neq 0$ this gives a solution isometric with

$$ds^2=(1-M/r)^2(dt^2-dr^2-r^2d\theta^2-r^2\sin^2\theta\, d\varphi^2)$$

which is the second unphysical solution discussed by Pavelle.[1]

(c) Finally in the 2×2 decomposable case we note that, if $ds^2=ds^{(1)2}+ds^{(2)2}$, coordinate systems exist such that

$$\begin{aligned} ds^{(1)2}&=e^{2\psi}[(dx^0)^2-(dx^1)^2],\\ ds^{(2)2}&=-e^{2\theta}[(dx^2)^2+(dx^3)^2], \end{aligned} \tag{9}$$

and their respective scalar curvatures $R^{(1)}$ and $R^{(2)}$ satisfy

$$\psi_{,00}-\psi_{,11}+\tfrac{1}{2}R^{(1)}e^{2\psi}=0,$$
$$\theta_{,22}+\theta_{,33}+\tfrac{1}{2}R^{(2)}e^{2\theta}=0.$$

For $R^{(1)}$ and $R^{(2)}$ constant, the general solutions of these equations are, respectively,

$$e^{-\psi}=1-\tfrac{1}{4}R^{(1)}[(x^0)^2-(x^1)^2],$$
$$e^{-\theta}=1+\tfrac{1}{4}R^{(2)}[(x^2)^2+(x^3)^2].$$

The only space-time of the above type satisfying $R_{ab}=0$ is of course flat space-time, and, as noted in theorem 6, Eq. (9) determines an Einstein space if and only if $R^{(1)}=R^{(2)}$. For $R^{(1)}+R^{(2)}\neq 0$, the resulting space-times have a type-***D*** Weyl tensor, and for $R^{(1)}+R^{(2)}=0$ they are conformally flat. An example of this latter type is given by the V_4 with line element (λ a constant)

$$ds^2=\lambda^2(x^1)^2(dx^0)^2+2\,dx^0dx^1-(dx^2)^2-\cos^2(\lambda x^2)(dx^3)^2. \tag{10}$$

Here $R^{(1)}=\lambda$ and $R^{(2)}=-\lambda$. The conformal flatness of this metric may be illustrated by the transformation

$$\lambda x^a=(t-r,\ r^{-1},\ \tfrac{1}{2}\pi-\theta,\ \varphi),$$

under which (10) becomes

$$ds^2=\lambda^{-2}r^{-2}(dt^2-dr^2-r^2d\theta^2-r^2\sin^2\theta\, d\varphi^2).$$

It could thus also have been obtained from (8) by choosing the constant $\alpha=0$.

This metric was first discovered by Robinson[6] and in the orthodox theory represents a spherically symmetric solution to the combined gravitational and electromagnetic field equations for empty space,

$$R_b{}^a+\varphi^{ac}\varphi_{cb}+{}^*\varphi^{ac}\,{}^*\varphi_{cb}=0,$$
$$\varphi^{ab}{}_{;b}={}^*\varphi^{ab}{}_{;b}=0,$$

where φ_{ab} is the electromagnetic field and ${}^*\varphi_{ab}$ is its dual.

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