Yang's Gravitational Field Equations*

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Yang's gravitational field equations *in vacuo* can be regarded as "derivative" equations of both Einstein's equations and Nordström's equations, and embrace all their solutions. Yang's equations admit monopole gravitational radiations; therefore no analog of the Birkhoff theorem can be valid. The most general static spherical-symmetric solution contains four arbitrary parameters. In particular, $ds^2 = -dt^2 + (1 + c_1/r + c_2 r^2)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta \times d\varphi^2)$ is a two-parameter exact solution. This metric possesses no gravitational red shifts.

Recently by using an integral formalism for gauge fields, Yang[1] has proposed the following gravitational field equations for pure spaces:

$$R_{ij;k} - R_{ik;j} = 0. \tag{1}$$

It is interesting to notice that (1) are satisfied by all vacuum solutions of Einstein's general relativity

$$G_{ij} + \Lambda g_{ij} = 0 \tag{2}$$

and Nordström's second theory[2-4] (in Einstein-Fokker form)

$$R = 0, \quad C_{ijkl} = 0, \tag{3}$$

where C_{ijkl} is the Weyl tensor. In fact, from Bianchi identities and the definition of the Weyl tensor, a simple calculation shows that (1) are equivalent to

$$G_{ij;k} - G_{ik;j} = 0, \tag{4}$$

or

$$R_{,i} = 0, \quad C_{ijkl;}{}^{l} = 0. \tag{5}$$

Equations (4) are differentiated equations[5] (antisymmetrized in j and k) of (2); while (5) are differentiated equations (summed over l) of (3). Hence Yang's theory is a derivative theory of both Einstein's theory and Nordström's theory. It is amusing and intriguing that the two eminent and structurally different theories of gravity could be embraced in a single set of equations.

Since Nordström's theory admits monopole radiations, Yang's theory admits them too. Therefore no analog of the Birkhoff theorem of general relativity could be valid for Yang's theory. More specifically, (1) have the following time-dependent spherical-symmetric solution (the time dependence cannot be transformed away by coordinate changes):

$$ds^2 = \left(\frac{c_0 t f(r-t)}{r} + \frac{g(r+t)}{r}\right)^2 (-dt^2 + dr^2 + r^2 d\Omega^2), \tag{6}$$

where c_0 is an arbitrary constant, and f and g are two arbitrary functions. Hence spherical symmetry does not imply time independence.[6]

In the static spherical-symmetric case, the metric can be put in the form

$$ds^2 = -e^{2\Phi(r)} dt^2 + e^{2\Lambda(r)} dr^2 + r^2(dr^2 + \sin^2\theta \, d\psi^2). \tag{7}$$

For this metric (1) reduce to

$$\Lambda'' - 2\Lambda'^2 + \Lambda'\Phi' + \Phi'^2 + r^{-2}(1 - e^{2\Lambda}) = 0 \tag{8}$$

and

$$\Phi''' + 2\Phi''\Phi' - 3\Phi''\Lambda' - 2\Lambda'\Phi'^2 - \Lambda''\Phi' + 2\Lambda'^2\Phi' + 2r^{-1}[\Phi'' - 2\Phi'\Lambda' - \Lambda'' + 2\Lambda'^2] - 2r^{-2}\Phi' - 2r^{-3}(1 - e^{2\Lambda}) = 0. \tag{9}$$

An integration of (9) gives

$$\Phi'' + \Phi'^2 - \Phi'\Lambda' + 2r^{-2}(\Phi' - \Lambda') + r^{-2}(1 - e^{2\Lambda}) + \tfrac{1}{2}R_0 e^{2\Lambda} = 0, \tag{10}$$

where R_0 is the constant scalar curvature. Both (8) and (10) are second order in Λ and Φ'; hence the general solution of (8) and (10) has four arbitrary parameters. (Because of nonlinearity there might be a discrete number of such sets of four parameters.) The additive constant in Φ can be removed by a change of time scale. Therefore the general static spherical-symmetric solution has four arbitrary parameters. This demonstrates that the solution of (1) is much richer than that of (2) (two parameters) or (3) (one parameter). As a matter of fact

$$ds^2 = -dt^2 + (1 + c_1/r + c_2 r^2)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) \tag{11}$$

is a solution of (1) but not of (2) or (3); (11) possesses no gravitational red shifts.[7] The problems of boundary conditions and sources for (1) deserve extensive studies to clear up this richness of solutions. In view of the present success of the renormalization of gauge theories, these studies could contribute to the solution of the long-standing problems of quantum gravity.

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[2]G. Nordström, Ann. Phys. (Leipzig) 42, 533 (1913), and 43, 1101 (1914).
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[4]W.-T. Ni, Astrophys. J. 176, 769 (1972).
[5]The second term in (2) is a constant term and, hence, does not contribute to (4).
[6]In view of this result, the recent claim of Thompson [Phys. Rev. Lett. 34, 507 (1975)] that the Birkhoff theorem of general relativity generalizes to (1) is invalid.
[7]W.-T. Ni, report presented at the annual meeting of the Chinese Physical Society, Chung-Li, Taiwan, 26 January 1975 (unpublished).

Geometrically Degenerate Solutions of the Kilmister-Yang Equations

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I solve, in principle, the Kilmister-Yang equations for the degenerate cases of conformal flatness and decomposability of space-time. The unphysical metrics discussed by Pavelle belong to these degenerate classes. Perhaps using the methods outlined here it will be possible to determine if these "unphysical" metrics are isolated examples or if they are typical of such "geometric-degenerate" classes.

In a recent paper Pavelle[1] has discussed certain solutions of the Kilmister-Yang (KY) equations[2,3] and has argued that in particular conformally flat solutions to these equations should not be allowable in the theory. Of course conformally flat solutions of the Einstein vacuum equations are necessarily flat and hence no additional constraint of this type is necessary there. Furthermore another unphysical solution discussed by Pavelle is also degenerate in the sense that it possesses a timelike parallel vector field and hence is decomposable. In fact almost all the solutions exhibited recently[1,4,5] possess some degenerate "geometric" property which is ruled out by the field equations in the orthodox theory (i.e., the only solutions with such properties are necessarily flat).

I present here some general theorems on conformally flat and decomposable spaces which should enable the construction of many solutions to the KY equations and hence a fuller discussion of the unphysical nature of the types of solu-