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[18]We neglect the interaction between charged droplets. This is believed to be small (except possibly very near T_c) because the charge screening length is less than the average interparticle spacing over the entire phase curve.

COMMENTS

Equivalence of a One-Dimensional Fermion Model and the Two-Dimensional Coulomb Gas

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We show that the one-dimensional Luttinger model generalized to include spin and backward scattering is equivalent to a two-dimensional Coulomb gas. Scaling equations are derived and correlation functions are given simple physical interpretation in terms of the Coulomb gas; e.g., existence of an energy gap can be understood in terms of Debye screening. We conclude that an energy gap exists for $U_\parallel < |U_\perp|$ so that triplet excitations are nondivergent, and we provide physical arguments to support the exponents proposed by Luther and Emery for singlet excitations for general coupling constant.

There has been considerable recent interest in an extension of the Luttinger model[1] to include spin as well as scattering from $+k_F$ to $-k_F$. The Hamiltonian is written as $H = H_S + H_L$, where H_S is the usual Luttinger-model Hamiltonian

$$H_S = v_F \sum_{k,s} k(a_{k,s}^\dagger a_{k,s} - b_{k,s}^\dagger b_{k,s}) + 2L^{-1} \sum_k V\rho_1(k)\rho_2(-k), \tag{1}$$

with $a_{k,s}$ ($b_{k,s}$) describing spin-$\frac{1}{2}$ fermions with momentum k ($-k$), and $\rho_1(k)$ and $\rho_2(k)$ density operators,

$$\rho_1(k) = 2^{-1/2} \sum_{p,s} a_{p+k,s}^\dagger a_{p,s}, \quad \rho_2(k) = 2^{-1/2} \sum_{p,s} b_{p+k,s}^\dagger b_{p,s}.$$

The large-momentum-transfer terms are described by

$$H_L = \sum_{s,s'} \int dx\, \Psi_{1,s}^\dagger(x)\Psi_{2,s'}^\dagger(x)\Psi_{1,s'}(x)\Psi_{2,s}(x)\,(U_\parallel \delta_{s,s'} + U_\perp \delta_{s,-s}), \tag{2}$$

where $\Psi_{1s}(x) = L^{-1/2}\sum_k \exp(ikx)a_{k,s}$ and $\Psi_{2s}(x) = L^{-1/2}\sum_k \exp(ikx)b_{k,s}$. Luther and Emery[2] (LE) have pointed out the similarity of this problem to the Kondo problem and have produced an exact solution of this model for a particular coupling constant $U_\parallel(2\pi v_F)^{-1} = -\frac{3}{5}$. They found an energy gap in the spin degrees of freedom and calculated exponents for the charge-density-wave response χ_S and singlet pairing response P_S. Their result on the spin-density-wave response χ_T and triplet pairing response P_T was found to be in error by one of us[3] who concluded that these triplet excitations are in fact nondivergent. This result is consistent with an exponentially activated uniform magnetic susceptibility χ_0. LE also argued on the basis of scaling that the gap exists for all $U_\parallel < 0$ and suggested exponents for arbitrary coupling constants. In this work we further exploit the similarity to the Kondo problem and show for general $U_\perp$ and $U_\parallel$ that the interacting-fermion problem at $T = 0$ is equivalent to a two-dimensional Coulomb gas at finite temperature. This problem has been studied in connection with the theory of melting in two dimensions[4] as well as the two-dimensional X-Y model.[5] On the basis of this equivalence we can derive scaling equations and produce physical arguments independent of the exact solution and scaling arguments to show that a gap exists for all $U_\parallel < |U_\perp|$. This in turn lends support to the exponents for singlet excitations suggested by LE for general coupling constant.

Our starting point is the observation by LE that H can be factorized into charge-density and spin-density components which commute with each other.[6] The charge-density component is the usual Luttinger model with an interaction constant $(2V - U_\parallel)$ which can be solved easily. We shall focus our attention only on the spin-density part,

$$H_\sigma = 2\pi v_F L^{-1}\sum_k [\sigma_1(k)\sigma_1(-k) + \sigma_2(-k)\sigma_2(k)] - L^{-1}\sum_k U_\parallel \sigma_1(k)\sigma_2(-k) + U_\perp(2\pi\alpha)^{-2}\int dx\{\exp(2^{1/2}[\varphi_1(x)+\varphi_2(x)]) + \text{c.c.}\}, \tag{3}$$

where

$$\varphi_j(x) = 2\pi L^{-1}\sum_k k^{-1}\exp(-\tfrac{1}{2}\alpha|k| - ikx)\sigma_j(k) \tag{4}$$

and $\sigma_1(k) = 2^{-1/2}\sum_p (a_{p+k\uparrow}{}^\dagger a_{p\uparrow} - a_{p+k\downarrow}{}^\dagger a_{p\downarrow})$. The boson representation has been used and Eq. (3) is rigorous only if the cutoff $\alpha \to 0$. However we follow LE and interpret $v_F\alpha^{-1}$ as a finite bandwidth. As do LE we perform a canonical transformation to eliminate the $U_\parallel$ term and thereby replace the factor in the exponent by $2^{1/2}e^\varphi[\varphi_1(x)+\varphi_2(x)]$, where $\tanh(2\varphi) = U_\parallel(2\pi v_F)^{-1}$. We now perform perturbation expansions in powers of $U_\perp$. As an example we consider $Z = \langle\exp(-\beta H_\sigma)\rangle$ which can be expanded as

$$Z = \sum_n [U_\perp/(2\pi\alpha)^2]^{2n}\int_0^\beta d\tau_{2n}\cdots\int_0^{\tau_3} d\tau_2\int_0^{\tau_2} d\tau_1 \times\int_0^L (\textstyle\prod_i dx_i)\sum_Q\langle\prod_i \exp\{s_Q(i)2^{1/2}e^\varphi[\varphi_1(\tau_i,x_i)+\varphi_2(\tau_i,x_i)]\}\rangle_{H_0}, \tag{5}$$

where $s_Q(i) = \pm 1$ depending on whether the $U_\perp$ term written out in Eq. (3) (flip-up term) or its complex conjugate (flip-down term) is chosen. Q denotes all possible such combinations. The average is taken over a noninteracting Hamiltonian. Each term in the Q sum can be factored into a product of two averages, each of which involves only either particle 1 or particle 2. Each average can be computed by moving all destruction operators to the right using repeatedly the identity $e^Ae^B = e^Be^Ae^{[A,B]}$ if $[A,B]$ is a c-number. Each commutation yields factors like

$$\exp(s_Q(i)s_Q(j)2e^{2\varphi}\ln\{[\alpha + v_F(\tau_j - \tau_i) \pm i(x_j - x_i)]/\alpha\})$$

for particle 2 or 1. Putting everything together we obtain

$$Z = \sum_n \frac{1}{(n!)^2}\left[\frac{U_\perp}{(2\pi\alpha)^2}\right]^{2n}\int(\prod_i^{2n} d^2r_i)\exp\left[\sum_{i>j} s_is_j 2e^{2\varphi}\ln\left(\frac{|\vec{r}_i - \vec{r}_j|^2}{\alpha^2}\right)\right], \tag{6}$$

where $\vec{r} = (x, v_F\tau)$ and the integration is over a two-dimensional space of area $v_F\beta L$. The sign s_i is positive for $i = 1$ to n and negative for $i = n+1$ to $2n$. If we introduce $\exp(-\bar\beta\mu) = U_\perp/(2\pi)^2$ and $\bar\beta q^2 = 2e^{2\varphi} = 2(1 + U_\parallel/2\pi v_F)^{1/2}(1 - U_\parallel/2\pi v_F)^{-1/2}$, Eq. (6) is the grand canonical ensemble partition function for a gas of charge q and $-q$ at temperature $T = \bar\beta^{-1}$ interacting via the two-dimensional Coulomb potential. In deriving Eq. (6) we have performed the sum over Q by allowing the particles to move everywhere in τ space and then dividing by $(1/n!)^2$ for overcounting. We also simulate the cutoff by stipulating that the particles are hard discs of radius α. Our approach is very similar to that used by Schotte[7] to show that the Kondo problem is equivalent to a one-dimensional charged gas of hard rods.[8,9]

Inspired by the work on the Kondo problem,[10] Kosterlitz[5] has studied the scaling properties of the two-dimensional Coulomb gas. By successively integrating out pairs that are spaced between α and $\alpha + d\alpha$ it can be shown to lowest order in $U_\perp{}^2$ that

$$Z = \tilde Z\exp\{2\pi v_F\beta L[U_\perp/(2\pi\alpha)^2]^2\alpha\, d\alpha\},$$

where $\tilde Z$ is the same as Eq. (6) except that the cutoff is $\alpha + d\alpha$ and $U_\parallel$ and $U_\perp$ are modified in the way given by the scaling equations

$$d(U_\parallel/2\pi) = -2(U_\perp/2\pi)^2\, d\ln\alpha, \tag{7}$$

$$d(U_\perp/2\pi) = -2(U_\parallel/2\pi)(U_\perp/2\pi)\, d\ln\alpha. \tag{8}$$

Equations (7) and (8) are the lowest-order expansion about the fixed point at $U_\perp = U_\parallel = 0$. While these equations can also be obtained by generalizing the work of Menyhard and Solyom[11] our approach does not involve making any *a priori* assumption concerning scaling.

The solution of Eqs. (7) and (8) is summarized in Fig. 1. It is obvious that $U_\parallel^2 - U_\perp^2$ is independent of α. For $U_\parallel \geq |U_\perp|$ scaling is towards $U_\perp = 0$ whereas for $U_\parallel < |U_\perp|$ scaling is towards stronger and stronger coupling. In terms of the Coulomb gas $U_\parallel = |U_\perp|$ defines a transition temperature T_c such that $U_\parallel \geq |U_\perp|$ implies $T \leq T_c$ and the positive and negative charges are bound together in pairs. Above T_c the gas begins to be ionized. For the isotropic case $U_\parallel = U_\perp \equiv g_1$, positive g_1 corresponds to a low-temperature gas of pairs while negative g_1 corresponds to a high-temperature ionized gas.

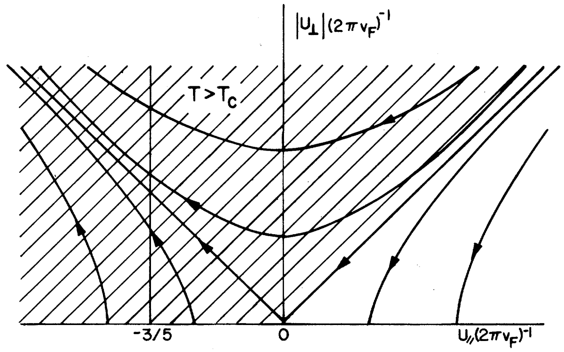

FIG. 1. Scaling diagram in the $U_\parallel$–$|U_\perp|$ space. Scaling is in the direction of the arrow for increasing cutoff α. In the Coulomb-gas analogy the shaded area $U_\parallel < |U_\perp|$ corresponds to temperatures above T_c, the ionization transition temperature. While this picture is based on first-order renormalization group it is expected to be qualitatively correct for strong coupling. We expect screening in the Coulomb gas and therefore an energy gap for the one-dimensional problem in the shaded area.

We can also consider the behavior of the susceptibilities. Using the boson representation LE have shown that each of the response functions can be factored into parts depending on the charge-density and spin-density degrees of freedom. As an example

$$\chi_T(x,\tau) = \langle \Psi_{2\downarrow}{}^\dagger \Psi_{1\uparrow}(x,\tau)\Psi_{1\uparrow}{}^\dagger \Psi_{2\downarrow}(0,0)\rangle = \chi_T{}^\rho(x,\tau)\tilde{\chi}_T(x,\tau), \tag{9}$$

where $\chi_T{}^\rho$ can be calculated by use of the Luttinger model. Defining similar quantities $\tilde{\chi}_S$, $\tilde{P}_S$, and $\tilde{P}_T$ for charge-density-wave and pairing responses we obtain

$$\tilde{P}_{S(T)}(x,\tau) = \tilde{\chi}_{S(T)}(x,\tau) = \langle \exp[\Psi_{S(T)}(x,\tau)]\exp[-\Psi_{S(T)}(0,0)]\rangle_{H_\sigma}, \tag{10}$$

where

$$\Psi_{S(T)}(x,\tau) = 2\pi L^{-1}\sum_k k^{-1}\exp(-\tfrac{1}{2}\alpha|k| - ikx)\{2^{-1/2}e^{\pm\varphi}[\sigma_1(k,\tau)\pm\sigma_2(k,\tau)]\}$$

and the + (−) spin goes with S (T). These correlation functions can again be translated into the Coulomb-gas picture by use of a perturbation expansion. The singlet response is given by $\tilde{\chi}_S(\vec{r}) = \langle\exp[-V(\vec{r},0)]\rangle$, where

$$V(\vec{r},\vec{r}\,') = \tfrac{1}{2}e^{2\varphi}\ln(|\vec{r}-\vec{r}\,'|/\alpha)^2 - \sum_i s_i e^{2\varphi}[\ln(|\vec{r}-\vec{r}_i|/\alpha)^2 - \ln(|\vec{r}\,'-\vec{r}_i|/\alpha)^2] \tag{11}$$

is the potential energy which corresponds to inserting two charges of $\pm\frac{1}{2}q$ into the Coulomb gas. The first term is the bare interaction between the extra charges which is screened by the second term describing interaction with other charges in the gas. We expect the screening to be insulatorlike for $T < T_c$ and metallic (exponential) for $T > T_c$. We can interpret $\tilde{\chi}_S(\vec{r})$ as the probability of finding the two charges separated by $\vec{r}$. Let us now make some physical argument based on our intuitive understanding of the Coulomb gas. We can write $\chi_S(\vec{r}) = \exp[-\beta H_{\mathrm{eff}}(\vec{r})]$, where $H_{\mathrm{eff}}(\vec{r})$ is the effective energy of the system when the charges are $\vec{r}$ apart. For $U_\parallel < U_\perp$ we have an ionized gas and we expect $H_{\mathrm{eff}}(\vec{r}) \propto \exp(-\kappa|\vec{r}|)$, where κ is the Debye screening given by $\kappa^2 = 4\pi\,\partial n/\partial\mu$. Replacing τ by it we then obtain $\tilde{\chi}_S(0,t) = \exp(-ce^{-i\kappa t})$, where c is some constant. Debye screening in the Coulomb gas then corresponds to a gap in the excitation spectrum. It is amusing that when κ is calculated with use of $n = \partial \ln Z/\partial(\beta\mu)$ to first order in $U_\perp^2$ we obtain $\kappa = 2\Delta$, where $\Delta = |U_\perp|(2\pi\alpha)^{-1}$ is the gap obtained by LE. Since $\tilde{\chi}_S \to 1$ for $t\to\infty$ we can argue that the low-frequency behavior of χ_S and P_S is given by the charge-density degrees of freedom and the exponents are those suggested by LE, i.e., $\chi_S = \omega^\mu$ and $P_S = \omega^{\mu'}$, where $\mu = -2 + (1-v)^{1/2}(1+v)^{-1/2}$, $\mu' = -2 + (1+v)^{1/2}(1-v)^{-1/2}$, and $v = (2V - U_\parallel)(2\pi v_F)^{-1}$.

Performing the same expansion for $\tilde{\chi}_T$ we obtain

$$\tilde{\chi}_T(\vec{r}) = \langle e^{i2\theta(\vec{r})}e^{-i2\theta(0)}\rangle_{H_\sigma}\exp[-e^{-2\varphi}\ln(r/\alpha)], \tag{12}$$

where $\theta(\vec{r}) = \sum_i s_i \tan^{-1}[(x_i - x)/v_F(\tau_i - \tau)]$. We note that the expression for $\tilde{\chi}_T$ is very different from that for $\tilde{\chi}_S$. In particular if $\theta(\vec{r})$ are uncorrelated for large $\vec{r}$, then $\tilde{\chi}_T(\vec{r})$ may approach zero for large $\vec{r}$. Indeed $\theta(\vec{r}) - \theta(\vec{r}')$ is the angle between the electric field vector at $\vec{r}$ and $\vec{r}'$ in the Coulomb gas. Furthermore, reference to Kosterlitz[5] shows that $\langle\exp\{i[\theta(\vec{r}) - \theta(0)]\}\rangle$ is proportional to the spin-spin correlation function in the two-dimensional X-Y model due to excitation of vortex pairs. Again $U_\parallel < |U_\perp|$ corresponds to $T > T_c$ and we expect the correlation function to decay exponentially. The behavior of $\tilde{\chi}_T(r, t)$ is then expected to be proportional to $\exp[i\kappa(t^2 - x^2)^{1/2}]$. This has been verified by explicit calculation[3] for the particular coupling constant where LE obtained exact solutions. The oscillatory behavior in $\tilde{\chi}_T(x, t)$ for large t implies an energy gap in $\tilde{\chi}_T(q, \omega)$ which in turn rules out any divergence in χ_T and P_T for small ω.

To conclude we find that the strong-coupling fermion problem corresponds to a Coulomb gas at high temperatures where we have a good physical understanding of the situation. On the basis of these physical pictures we can obtain the behavior of the singlet and triplet excitations without using scaling or knowledge about the exact solution. On the other hand once we are above the ionization transition we do not expect additional unstable fixed points to exist as these would imply further phase transitions in the Coulomb gas. If additional unstable fixed points do not exist, LE's suggestion of scaling onto their exact solution will be justified. We expect that the qualitative behavior is given correctly and that the exponents for singlet excitation can be calculated with use of only the charge-density-wave component. We should mention that first-order renormalization-group equations for $\tilde{\chi}_{S(T)}$ can be derived.[5] Using these we generalized Solyom's[11,12] exponents to the anisotropic case for $U_\parallel > U_\perp$ while for $U_\parallel < |U_\perp|$ we have confirmed that the gap remains the most important feature. Further details will be given in a future publication.

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