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# Critical Behavior at the Onset of $\vec{k}$-Space Instability on the λ Line

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(Received 28 July 1975)

We calculate the critical behavior of systems having a multicritical point of a new type, hereafter called a Lifshitz point, which separates ordered phases with $\vec{k}=0$ and $\vec{k}\neq 0$ along the λ line. For anisotropic systems, the correlation function is described in terms of four critical exponents, whereas for isotropic systems two exponents suffice. Critical exponents are calculated using an ϵ-type expansion.

We introduce a new multicritical point whose critical behavior is strikingly different from any reported previously. Among other attributes, it is, in general, necessary to introduce a pair of exponents to replace each of the correlation-function critical exponents $\eta$ and $\nu$. To introduce this new multicritical point, consider for simplicity the bare free-energy-density functional for an isotropic system described by a scalar order parameter,[1]

$$F(M) = a_2M^2 + a_4M^4 + a_6M^6 + \dots + c_1(\nabla M)^2 + c_2(\nabla^2 M)^2 + \dots . \quad (1)$$

In the vicinity of the usual critical point, the only terms pertinent to a study of the system's critical behavior are those with the coefficients $a_2$, $a_4$, and $c_1$.[2] The special point we shall introduce is characterized by the necessity to consider also the $c_2$ term. We shall refer to this as a Lifshitz point.[3] It will occur whenever $c_1$ or its renormalized counterpart vanishes at the phase transition. The Lifshitz point is thus analogous to a tricritical point where the $a_6$ term must be considered because of the vanishing of $a_4$ or its renormalized counterpart.[4] Here we discuss the critical behavior of a system in the vicinity of a Lifshitz point using renormalization-group techniques.

An example of a phase diagram describing a system exhibiting a Lifshitz point is shown in Fig. 1. As one moves from 1 to 2 along the λ line of second-order phase transitions by varying a parameter $P$, the ordered state changes from ferromagnetic to helicoidal.[5] This could be achieved, for example, by varying the pressure

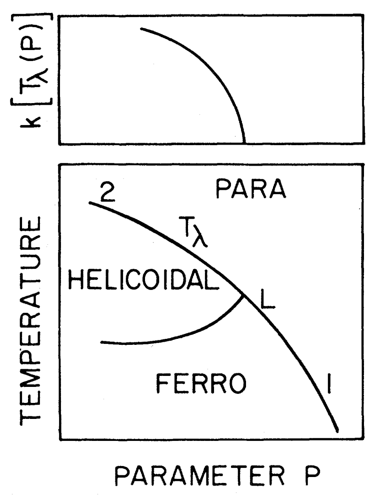


FIG. 1. Schematic phase diagram describing a magnetic system exhibiting a Lifshitz point $L$. The curve 1-2 is the λ line of second-order phase transitions. The upper half of the figure shows the equilibrium wave vector $\vec{k}$ of the helicoidal and ferromagnetic phases on the λ line.

or the composition of mixed compounds or alloys. Note that the wave vector $\vec{k}$ characterizing the helicoidal structure increases *continously* from $\vec{k}=0$ at the Lifshitz point $L$. In terms of (1), the coefficient $c_1$ changes sign and $c_2$ is positive at $L$. Simple models which display a $\lambda$ line of this type have been studied in the molecular-field approximation by Yoshimori,[6] Villain,[7] and Kaplan.[8]

Before describing the renormalization-group treatment, it is useful to apply the Ginsburg criterion so as to determine the critical dimensionality $d_0$ above which classical behavior is to be expected.[9] Consider a Landau free-energy-density functional of the form

$$F(\vec{M}) = a_2' tM^2 + a_4 M^4 + c_\beta (\nabla_\beta \vec{M})^2 + c_\alpha (\nabla_\alpha{}^2 \vec{M})^2 + c_\gamma (\nabla_\alpha{}^2 \vec{M})\cdot(\nabla_\beta{}^2 \vec{M}). \tag{2}$$

Here $\nabla_\alpha$ and $\nabla_\beta$ are gradient operators of dimensionality $m$ and $d-m$, respectively, $\vec{M}$ is an $n$-component vector order parameter, and $t$ is the reduced temperature. We are thus assuming that for only $d-m$ coordinates is there a contribution to $F$ which is quadratic in the spatial derivatives of $\vec{M}$. For simplicity, we have assumed that the system is otherwise isotropic.[10] In view of the isotropy of the two sets of spatial coordinates, we can regard both $m$ and $d$ as continuous variables (with $m \le d$) and calculate $d_0(m)$, the critical dimensionality above which the critical exponents adopt classical values.[11] The quantity $d_0(m)$ is most simply determined by requiring that the contributions to the equilibrium free energy from the terms $a_2'tM^2$ and $a_4M^4$ be of the same power of $t$. For $t>0$, this is equivalent to requiring that $a_4\langle M^2\rangle \propto a_4\int d^dk\langle |M_k|^2\rangle$ be of order $t$. It is then straightforward to show that $\langle M^2\rangle \propto t^{(d-2-m/2)/2}$, and the critical dimensionality is thus given by[12]

$$d_0(m) = 4 + m/2, \quad m \le 8. \tag{3}$$

The regions of $(d,m)$ space in which classical and nonclassical critical behavior will occur are thus separated by a critical-dimensionality line. This suggests that one might calculate critical exponents for a system associated with a point $(d,m)$ in the nonclassical region by expanding in a double power series in $\epsilon_\alpha = m_0 - m$ and $\epsilon_\beta = (d_0 - m_0) - (d-m)$, where $(d_0, m_0)$ is any point on the critical-dimensionality line. This constitutes a generalization of the usual Wilson-Fisher $\epsilon$ expansion.[2]

Turning now to the renormalization-group calculation, we consider a Landau-Wilson Hamiltonian of the form

$$\mathcal{H} = -\tfrac{1}{2}\int_q v(\vec{q})\vec{\sigma}_q\cdot\vec{\sigma}_{-q} - u\int_q\int_{q'}\int_{q''}(\vec{\sigma}_q\cdot\vec{\sigma}_{q'})(\vec{\sigma}_{q''}\cdot\vec{\sigma}_{-q-q'-q''}), \tag{4a}$$

$$v(\vec{q}) = r + c_1 q_\alpha{}^2 + q_\beta{}^2 + (q_\alpha{}^2)^2 + c_\gamma q_\alpha{}^2 q_\beta{}^2 + c_\delta (q_\beta{}^2)^2 + O(q^6), \tag{4b}$$

$$q_\alpha{}^2 = \sum_{i=1}^{m} q_i{}^2, \quad q_\beta{}^2 = \sum_{i=m+1}^{d} q_i{}^2. \tag{4c}$$

Note that we have normalized the coefficients of both $q_\beta{}^2$ and $q_\alpha{}^4$ to unity. To first order in $\epsilon_\alpha$ and $\epsilon_\beta$ the renormalization gives

$$v'(q) = \zeta^2 a^{-m} b^{-(d-m)}[r + q_\beta{}^2/b^2 + c_1 q_\alpha{}^2/a^2 + (q_\alpha{}^2)^2/a^4 + c_\delta (q_\beta{}^2)^2/b^4 + c_\gamma q_\alpha{}^2 q_\beta{}^2/a^2b^2 - 4(n+2)uA(r)], \tag{5a}$$

$$u' = \zeta^4 a^{-3m} b^{-3(d-m)}[u - 4(n+8)u^2 C(0)], \tag{5b}$$

where

$$A(r) = (2\pi)^{-d}\int[(q_\alpha{}^2)^2 + q_\beta{}^2 + r]^{-1} d^d q \tag{5c}$$

and $C(0) = dA/dr|_{r=0}$. The integration is over the volume $1/a \le |q_\alpha| \le 1$, $1/b \le |q_\beta| \le 1$, where $a^{-1}$ and $b^{-1}$ are small-momentum cutoffs. To describe critical behavior in the vicinity of the Lifshitz point it is necessary to introduce a pair of critical exponents to replace each of the critical exponents $\eta$ and $\nu$.[2] The subscripts $l2$ and $l4$ will be used to distinguish between them. In order to restore the coefficients of both $q_\beta{}^2$ and $q_\alpha{}^4$ to unity, we set

$$\zeta^2 = a^m b^{d-m} b^{2-\eta_{l2}}, \tag{6a}$$

$$a^{4-\eta_{l4}} = b^{2-\eta_{l2}}. \tag{6b}$$

As usual, $\eta_{l2}, \eta_{l4} = 0 + O(\epsilon^2)$. It is now easily seen that $c_\gamma$ and $c_\delta$ are irrelevant. The usual critical points are obtained for $c_1{}^* = \infty$. The special point we seek is obtained from the following fixed-point relations: Lifshitz-Gaussian,

$$c_1^* = u^* = 0; \tag{7a}$$

Lifshitz-Heisenberg,

$$c_1^* = 0, \quad u^* = \epsilon_l [4(n+8)C(0)]^{-1} \ln b, \tag{7b}$$

where

$$\epsilon_l = 4 - d + m/2 = \epsilon_\alpha/2 + \epsilon_\beta. \tag{7c}$$

The Gaussian fixed point is stable for $\epsilon_l < 0$ or $d > 4 + m/2$ and the Heisenberg fixed point is stable for $\epsilon_l > 0$.

The critical exponents $\nu$ are found in the usual way from

$$r_{s+1} - r^* = \sigma^{1/\nu}(r_s - r^*), \quad \sigma = a, b. \tag{8}$$

Taking $\sigma = a$ yields $\nu_{l4}$, the two-spin correlation-length exponent for spins joined by a vector whose components lie entirely in $\alpha$. Similarly, by setting $\sigma = b$, we obtain $\nu_{l2}$. In general there will be a crossover from $\nu_{l4}$ to $\nu_{l2}$ as the Lifshitz point is approached. Using (7b), (7c), and (8), we obtain for $\epsilon_l > 0$

$$\nu_{l4} = \frac{\nu_{l2}}{2} = \frac{1}{4}\left(1 + \frac{(n+2)}{2(n+8)}\epsilon_l\right) + O(\epsilon_l^2). \tag{9}$$

For the completely isotropic ($m = d$) model, the results of a second-order $\epsilon$ calculation for the Lifshitz-Heisenberg fixed point are

$$\eta_{l4} = -\frac{3}{20}\frac{(n+2)}{(n+8)^2}\epsilon_\alpha^2 + O(\epsilon_\alpha^3), \tag{10a}$$

$$\nu_{l4} = \frac{1}{4} + \frac{(n+2)}{16(n+8)}\epsilon_\alpha + \frac{(n+2)(15n^2+89n+4)}{960(n+8)^3}\epsilon_\alpha^2 + O(\epsilon_\alpha^3). \tag{10b}$$

Note that to second order in $\epsilon$ the Heisenberg-Lifshitz fixed-point relation $c_1^* = 0$ becomes $c_1^* - G(u^*)^2 = 0$ where $G < 0$ is the (cutoff-dependent) part of a Feynman diagram proportional to $q_\alpha^2$. This relation is analogous to that found by Stephen and McCauley.[4]

The new scaling relations appropriate to a general (nonisotropic) Lifshitz point are

$$m\nu_{l4} + (d-m)\nu_{l2} = 2 - \alpha_l, \tag{11a}$$

$$\gamma_l = (4 - \eta_{l4})\nu_{l4} = (2 - \eta_{l2})\nu_{l2}. \tag{11b}$$

The factor $4 - \eta_{l4}$ arises since, in the vicinity of the Lifshitz point, the order-parameter fluctuations in the $m$-dimensional subspace are dominated by the term $(q_\alpha^2)^2$ in Eq. (6b). The three-exponent scaling relations

$$\beta_l = \tfrac{1}{2}(2 - \alpha_l - \gamma_l), \quad \gamma_l = \beta_l(\delta_l - 1), \tag{12}$$

remain unchanged. Note that the right-hand equality in (11b) reduces the number of independent exponents from four to three for a general Lifshitz point. For an isotropic Lifshitz point there are two independent exponents.

The preliminary renormalization-group results given here are probably not sufficient to calculate the critical exponents accurately. In particular, since our second-order results were obtained by expanding from eight dimensions, it is too much to expect that they will be valid in three dimensions. The requirement that $\beta_l$ be positive implies, from Eqs. (11) and (12), that $\eta_{l4} > 1$ for the isotropic Lifshitz point when $d = 3$. A renormalization treatment for the anisotropic model to second order in $\epsilon$ will be given elsewhere.[13]

Inspection of Fig. 1 suggests that we define a *new critical exponent*. Approaching the Lifshitz point on the helicoidal segment of the $\lambda$ line we expect that the wave vector $\vec{k}$ will be related to the reduced variable $p = (P - P_c)/P_c$ by

$$k \sim |p|^{\beta_k}, \quad p < 0. \tag{13}$$

In a mean-field or Landau theory, $\beta_k = \frac{1}{2}$. Fluctuation effects will alter this value of $\beta_k$, as well as give rise to singularities in the shape of the phase boundaries at the Lifshitz point. Similar features are displayed at bicritical points.[14] A preliminary renormalization-group study indicates that $\beta_k$ cannot be expressed solely in terms of the Lifshitz-point critical exponents. To first order in $\epsilon_l$ there is no correction to the mean-field value of $\beta_k$.

In conclusion, we have shown that critical behavior at the Lifshitz point should be strikingly different from any previously reported. The conditions necessary to observe this point can be achieved in at least two ways: by varying an external parameter (e.g., pressure) and/or by preparing mixed compounds or alloys. The recent review by Cox[15] indicates that there exist a large number of systems in which a Lifshitz point may occur. Of particular interest is the $UAs_{1-x}S_x$ system whose phase diagram, as studied by Lander, Mueller, and Reddy,[16] shows transitions from ferromagnetic to sinusoidal to antiferromagnetic ordering as one moves along the $\lambda$ line by varying $x$.

We thank Professor S. Alexander for helpful discussions and Dr. D. Mukamel for comments on

the manuscript.

*Work supported in part by the Commission for Basic Research of the Israel Academy of Sciences and Humanities.

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[10]The system anisotropy will, in general, affect other terms in (2) [and also in (5)]. This will be discussed elsewhere.

[11]The classical correlation-function exponents for a Lifshitz point are $\nu_{l4}=\frac{1}{4}$, $\nu_{l2}=\frac{1}{2}$, $\eta_{l4}=\eta_{l2}=0$.

[12]Alternatively, Eq. (4) can be obtained by examining the divergence of the lowest-order diagrams contributing to the four-point function.

[13]To first order in $\epsilon_l$ the critical exponent for a uniaxial Ising-type Lifshitz point (i.e., $d=3$, $m=n=1$) are $\eta_{l4}=0$, $\nu_{l4}=\nu_{l2}/2=0.31$, $\alpha_l=\beta_l=0.25$, $\gamma_l=1.25$, $\delta_l=5.0$.

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# Muon-Pair Separation Measurements and Comparison with Transverse-Momentum Models*

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Rates of high-energy cosmic-ray muon pairs have been measured for separations up to 70 m. Detailed calculations imply that the mean transverse momentum $\langle p_T \rangle$ of mesons with $x > 0.01$ is $0.66 \pm 0.10$ GeV/$c$ at laboratory energies of $\gtrsim 10\,000$ GeV. We find that the high-$p_T$ muons result mostly from decay of abundantly produced particles with lifetimes $\gtrsim 10^{-8}$ sec, such as pions and kaons.

We report here measurements of pair separation distributions (decoherence curves) for deep underground muons using a main detector and auxiliary "outrigger" detectors. These data are compared to predictions of Feynman scaling and several $p_T$ models. Previous Utah decoherence data from Coats *et al.*[1] (analyzed by Adcock *et al.*[2] with a different interaction model than used here) had a significant systematic error because of the loss of about a 20% contribution to main-detector decoherence curves due to events with too large a number of muons in the main detector. An ex-