Charged-Particle Orbits in Sheared Magnetic Fields; Implications to Diffusion*

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Charged-particle motion across a sheared magnetic field and electrostatic wave is studied. Resonant-particle orbit widths are derived; two parametric dependences are found in the limits that the particle is near or far from the $k_{\|}=0$ plane, with a smooth transition between the two. The orbit widths are used to calculate diffusion coefficients in tokamak-type configurations. These predictions are confirmed by test-particle computer simulations in (x, y, v_x, v_y, v_z) coordinates employing a Monte Carlo collision operator.

At present there exists an order-of-magnitude discrepancy between the electron thermal diffusion observed experimentally and the smaller diffusion predicted by the neoclassical theory.[1] Pogutse[2] has suggested that resonant trapping by an electrostatic wave, rather than trapping in the nonuniform magnetic field, is responsible for the enhanced electron thermal diffusion. Taking the cross-field displacement of electrons trapped in the wave as the elementary step of a random-walk process, occurring in the presence of collisions, he arrived at a thermal diffusion coefficient which agreed more closely with experiments.

However, it was noted, especially by Brambilla and Lichtenberg[3] (BL), that shear in the magnetic field might significantly limit the efficiency of the mechanism proposed by Pogutse. BL derived an expression for the orbit width of particles with parallel velocities greater than the trapping velocity of the electrostatic potential. They used a Hamiltonian formalism to describe the motion of a particle in tokamak-type fields in the presence of a zero-frequency electrostatic wave and showed that the wave potential combined with the toroidal magnetic field created a series of drift orbits which diminish in amplitude as the inverse powers of the aspect ratio. For a large-aspect-ratio torus, the leading term, corresponding to a cylindrical configuration with toroidal periodicity and shear, is of primary importance. The restriction on particle velocity and wave frequency imposed by BL allowed one to determine the influence of shear on orbit size for only a limited parameter range.

This Letter presents a description of particle orbits which allows for particles moving with arbitrary velocities in a sheared magnetic field and an electrostatic potential of the form $\varphi=\varphi(\vec{k}\cdot\vec{r}-\omega t)\equiv\varphi(\theta)$. As done by Pogutse[2] the cylindrical configuration is approximated by a slab geometry.

Assuming $\omega \ll \Omega = eB/m$ and $k_\perp\rho = k_\perp v_\perp/\Omega \ll 1$, we can use the guiding-center equation of motion

$$\frac{d\vec{r}}{dt}=\frac{\vec{B}\times\vec{\nabla}\varphi}{B^2}+v_{\|}\frac{\vec{B}}{|\vec{B}|}, \tag{1}$$

$$\frac{dv_{\|}}{dt}=\frac{-e}{m}\frac{\partial\varphi}{\partial r_{\|}}. \tag{2}$$

We approximate the field in a rectangular coordinate system by $\vec{B}=[0, b(x), B_z]$ with $b(x)/B \ll 1$ and $B=|\vec{B}|$. $\vec{k}$ is taken to lie along the y axis, and subscripts $\|$ and $\perp$ refer to the parallel and perpendicular directions with respect to $\vec{B}$. Shear in $\vec{B}$ is modeled by expanding $k_{\|}(x)$ about $x=x_0$, the initial x coordinate of the particle, as

$$k_{\|}(x)/k_\perp=\alpha+\theta_s(x-x_0), \tag{3}$$

where $\alpha\equiv k_{\|}(x_0)/k_\perp$. With some algebraic manipulations Eqs. (1) and (2) may be reduced to relations among the differentials dx, $dv_{\|}$, and $d\varphi$. These relations were integrated to obtain a quartic equation for the half-width of the particle orbit, Δx:

$$(\alpha v_{\|0}-\omega/k_\perp)\Delta x+\tfrac{1}{2}(v_{\|0}\theta_s+\Omega\alpha^2)\Delta x^2 + \tfrac{1}{2}\Omega\alpha\theta_s\Delta x^3+\tfrac{1}{8}\Omega\theta_s^2\Delta x^4=(-e/m)\Delta\varphi/\Omega, \tag{4}$$

where $v_{\|0}$ is the initial parallel velocity of the particle and $\Delta\varphi$ is the change in potential over its orbit. When $\theta_s=0$, Eq. (4) reduces to the expression obtained by Pogutse[2]; while when $\omega=\alpha=0$ and $v_{\|0}\gg(|e|\Delta\varphi/m)^{1/2}$, Eq. (4) reduces to the results of BL.[4]

Considering only particles satisfying the resonance condition $k_{\|}(x_0)v_{\|0}=\omega$, as these particles will diffuse most rapidly, Eq. (4) reduces to

$$(\alpha^2+\beta^2)\Delta x^2+\alpha\theta_s\Delta x^3+\tfrac{1}{4}\theta_s^2\Delta x^4 = -2e\Delta\varphi/m\Omega^2, \tag{5}$$

where $\beta^2\equiv v_{\|0}\theta_s/\Omega$. We have solved Eq. (5) numerically to determine the resonant particle orbit amplitudes as a function of the change in potential. To obtain a better understanding of Eq. (5), its behavior is examined in the regimes where either shear or the change of the parallel velocity (due to potential) limits the cross-field excursion. The important parameter is

$$\frac{\alpha^2}{\beta^2}=\left(\frac{k_{\parallel}(x_0)}{k_{\perp}}\right)^2\frac{\Omega}{v_{\parallel 0}}\frac{1}{\theta_s}.$$

Taking $1/\theta_s=L_s$, the shear length, and defining a distance $s\equiv L_s[k_{\parallel}(x_0)/k_{\perp}]$, the distance of the resonance plane from the plane where $\vec{k}\cdot\vec{B}=0$, then we have $\alpha^2/\beta^2=s^2/(L_s v_{\parallel 0}/\Omega)$, where $v_{\parallel 0}/\Omega$ is of the order of a gyroradius. Setting $\Delta\varphi=\varphi_0$, with φ_0 a characteristic wave amplitude, we obtain in the limit $\alpha/\beta\gg 1$ that

$$\Delta x=\Delta x_p+\Delta x_1 \tag{6}$$

with

$$\Delta x_p=\pm\,\alpha^{-1}\Omega^{-1}(2|e|\varphi_0/m)^{1/2}, \tag{7}$$

which is the maximum drift amplitude obtained by Pogutse, and

$$\Delta x_1=(-\,\theta_s\Delta x_p/2\alpha^2\Omega)(v_{\parallel 0}+\alpha\Omega\Delta x_p). \tag{8}$$

In the limit $\alpha/\beta\ll 1$, we obtain the result of BL generalized to finite ω:

$$\Delta x_{\rm BL}=\pm(2|e|\varphi_0/m\Omega\theta_s v_{\parallel 0})^{1/2}. \tag{9}$$

In both limits shear reduces the size of the orbit, and hence the diffusion.

In the two limits there are quite different parametric dependences which can have important consequences for diffusion. Considering a stochastic scattering description for diffusion, i.e. $D_{\perp}\sim\Delta x^2/\Delta t$, where Δt is the mean step time, we readily see that in the Pogutse limit $D_{\perp}\sim(k_{\perp}/k_{\parallel}B)^2$. For a toroidal device we approximate $k_{\perp}/k_{\parallel}$ by mqR/a, where m is the mode number, $q=aB_t/RB_p$ is the safety factor with B_t and B_p the toroidal and poloidal magnetic fields, respectively, and R and a the major and minor radii. Thus the Pogutse diffusion is strongly dependent on mode number ($D_{\perp}\propto m^2$) and classical in nature ($D_{\perp}\propto B^{-2}$). On the other hand, in the BL limit [Eq. (9)], the diffusion becomes Bohm-like ($D_{\perp}\propto B^{-1}$) with dependence also on shear and parallel velocity. Keeping all dependences, Fig. 1 shows Δx obtained by solving Eq. (5) numerically. As is clear from this figure, Δx extends smoothly from one limit to the other, fitting the analytic approximations of Eqs. (6)–(9). We also note from Fig. 1 the obvious but important fact that Δx is limited,

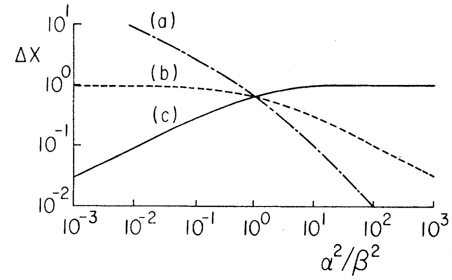

FIG. 1. The three curves shown illustrate the variation of the half-width of election orbits in the $\vec{E}\times\vec{B}$ direction, Δx, versus $\alpha^2/\beta^2=[k_{\parallel}(x_0)/k_{\perp}]^2\Omega L_s/v_{\parallel 0}$. Each plot is obtained by varying one system parameter, $[k_{\parallel}(x_0)/k_{\perp}]^2$, Ω, or L_s, while holding the others constant. The units are arbitrarily normalized by setting $k_{\perp}=1$ and $v_{\parallel 0}=0.01$. $e\varphi_0/m=5\times 10^{-6}$ throughout. (a) Δx versus Ω for $L_s=10^3$, $k_{\parallel}(x_0)/k_{\perp}=0.003$. (b) Δx versus $[k_{\parallel}(x_0)/k_{\perp}]^2\times 10^{-5}$ for $L_s=10^3$, $\Omega=1$. (c) Δx versus $L_s\times 10^{-3}$ for $\Omega=1$, $k_{\parallel}(x_0)/k_{\perp}=0.003$.

approximately, to the predictions of either Eq. (6) or (9), depending on which gives the *smaller* value. Thus for a given physical case, we can estimate Δx from the smaller of the two limit formulas.

In either the case for which the velocity changes limit the orbit amplitude as in Eq. (6), or that for which the shear limits the amplitude as in Eq. (9), the diffusion may be classified according to the magnitude of the collision frequency. In applying our results we distinguish between two important regimes: (a) the "football" regime for which $\nu_{\rm eff}=T\nu/e\varphi<\omega_B$, where ν is the collision frequency; (b) the plateau regime for which $\nu_{\rm eff}>\omega_B$. $\nu_{\rm eff}$ is the effective collision frequency for scattering out of the potential well. ω_B is the bounce frequency found by linearizing the equations of motion for small excursions of θ and x about a fixed point of the motion, and thereby obtaining an equation describing simple harmonic motion, with frequency

$$\omega_B=k_{\perp}[(\alpha^2+\beta^2)|e|\varphi_0/m]^{1/2}. \tag{10}$$

To examine the diffusion rates, for a given toroidal device, it is necessary to determine which is the appropriate regime.

We model the shear by $\theta_s\approx 1/Rq$, and take $k_{\perp}\approx m/a$ and $k_{\parallel}=O(1/Rq)$. The condition for shear to be the amplitude-limiting factor ($\alpha/\beta\ll 1$) becomes, for electrons, $m^2>a^2/\rho_e Rq$. If one assumes that the fastest-growing modes have $m\simeq a/\rho_i$, then this inequality is marginally satisfied for present-day tokamaks and well satisfied for fusion reactors, indicating that shear will limit the orbit amplitude and diffusion.

If the diffusion is occurring in the "football" regime, then the condition $\nu_{\text{eff}} < \omega_B$, considered in the regime for which shear is unimportant, reduces with the help of Eq. (10) to $(e\varphi/T)^{3/2}\lambda/Rq > 1$, where λ is the electron mean free path. For typical present-day tokamak parameters, $10^{13} \leq n \leq 5\times 10^{13}$ cm^{-3}, $1 \leq T \leq 2$ keV, and $Rq \simeq 2$ m, we find that for the highest density and lowest temperature (smallest λ), $e\varphi/T \geq 0.5$ is in the "football" regime; while for the largest λ this inequality becomes $e\varphi/T \geq 0.007$. Below these values of $e\varphi/T$ the plateau regime would be the appropriate one.

We now wish to be more explicit about the factors governing the particle diffusion and to display a particle simulation that verifies the dependences of these factors. The diffusion coefficient for stochastic scattering is given as usual by $D_\perp = \langle \Delta x^2 \rangle \nu_{\text{eff}} f_T$, where $\langle \Delta x^2 \rangle$ is the appropriate average over drift-orbit amplitude, and $f_T = (e\varphi/T)^{1/2}$ is the fraction of particles which are trapped. For the shearless case $D_\perp$ reduces to the simple estimate of Pogutse,[2]

$$D_\perp^F \approx (e\varphi/T)^{1/2}\rho^2(k_\perp/k_\parallel)^2\nu. \tag{11}$$

As was emphasized by Pogutse this equation is essentially of the form of the pseudoclassical formula[1] and for $k_\perp \gg k_\parallel$ the diffusion might be substantially enhanced.

In the plateau regime, the particle is likely to suffer a collision during the time of traversal of a drift orbit. Estimating the probability that a trapped particle completes its orbit before suffering a collision to be $(\omega_B/\nu_{\text{eff}})$, we obtain $D_\perp = \langle \Delta x \rangle^2 \omega_B f_T$. For the shearless case this reduces to

$$D_\perp^P \approx (e\varphi/T)^2\rho^2(k_\perp/k_\parallel)^2 k_\parallel v_T. \tag{12}$$

This regime is mentioned for completeness. Implicit in our derivation of Eq. (11) is the assumption that resonant particles dominate the diffusion process. To test this assumption as well as the parametric dependences of this equation we used a guiding-center code to simulate a Lorentz plasma with a Maxwellian distribution of particle velocities in a shearless magnetic field and electrostatic potential. This code[5] follows the motion of hundreds (typically 512) of noninteracting "test" particles in two spatial and three velocity dimensions subject to the externally applied $\vec{E}$ and $\vec{B}$ and to a collision operator[6] which simulates the three-dimensional velocity scattering of electrons by stationary ions. The perpendicular diffusion coefficient is obtained from the mean square deflection of particles across $\vec{B}$ divided by twice the time. Keeping all parameters, except one, fixed, the dependence of $D_\perp$ on $e\varphi/T$, Ω, and $k_\perp/k_\parallel$ predicted by Pogutse,[2] Eq. (11), has been verified by this code. Table I illustrates this, listing the simulation results for the perpendicular diffusion coefficient, $D_\perp^*$, for the parameters indicated. $D_\perp^*$ reached an asymptotic value in a time $\sim 1/\nu$ with an experimental spread as given in the table caption [the spread scaled roughly as (number of particles)$^{-1/2}$]. The $D_\perp^*$'s were normalized using a representative run. The attempt to equate the normalization coefficient with that derived analytically by Pogutse is not appropriate because of the different approximations made in each case. (For example, Pogutse considers only electron-electron collisions.) The results confirm the parametric dependence of the diffusion coefficient given by Eq. (11).

TABLE I. Results of computer-simulation parameter studies of the cross-field diffusion coefficient, $D_\perp^*$, listed for representative runs in the shearless case, "football" regime, and compared with the theoretically predicted values from Eq. (11) normalized using a typical case, $D_\perp^{F(n)}$. The normalizing runs are indicated by "(*n*)." The units used are arbitrarily set by choosing the system length in the direction of $\vec{E}$, $L = 2\pi$, and the plasma frequency, $\omega_{pe} = 1$. Periodic boundary conditions are used. In all examples $e/m = 1$ and for the potential, φ, $k = 1$, $\omega = 0$. The parameters varied are given explicitly. In addition, in runs 1–4, $v_t = 0.012\,125$, $\nu_{\text{coll}} = 10^{-5}$; in 5 and 6, $v_t = 0.001\,212\,5$, $\nu_{\text{coll}} = 10^{-7}$. The spread in the experimental data for $D_\perp^*$ is 4–8% for 1–4 and 8–12% for 5 and 6. The higher $e\varphi/T$ of runs 1–4 were chosen to allow larger ν_{coll}. More accurate determination of $D_\perp^*$ could be made in these cases. In all the simulations, the agreement with theory is well within the experimental spread in the data.

Run	$k_\perp/k_\parallel$	Ω/ω_{pe}	$e\varphi/T$	$D_\perp^*[(L/2\pi)^2\omega_{pe}]^{-1}$	$D_\perp^*/D_\perp^{F(n)}$
1	10	0.1	0.25	6.0×10^{-6}	1.0(*n*)
2	10	5.0	0.25	2.4×10^{-9}	1.0
3	5	1.0	0.25	1.5×10^{-8}	1.0
4	30	1.0	0.25	5.4×10^{-7}	1.0
5	10	0.1	0.04	9.0×10^{-11}	1.0(*n*)
6	10	0.1	0.09	1.45×10^{-10}	1.1

We have derived and solved a quartic equation for the half-width of particle orbits in the presence of a weakly sheared magnetic field and of an electrostatic wave. By examining the quartic equation we have shown that there are two competing mechanisms which change the phase of the electric field seen by the particle so as to limit the size of the drift orbits. One arises from the change in velocity along the field lines (Pogutse mechanism). The second arises from the shear (BL mechanism). A criterion is found to determine which mechanism predominates. Orbit sizes obtained from the quartic equations and particle simulations agree with the above conclusions. For the case without shear, particle simulation of the diffusion substantiates the predicted parametric dependence of the perpendicular diffusion coefficient.

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