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Low-Frequency Stability of Astron Configurations*

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A sufficient condition is obtained for stability of low-frequency kink perturbations of an astron-type particle ring embedded in a dense low-temperature plasma.

There is renewed interest in the use of intense beams of high-velocity particles in toroidal astron configurations.[1-3] Such beams may allow the confinement and heating of a plasma at fusion temperatures and densities. It is desired to have the directed kinetic energy density of the particle beam as large a fraction (β) as possible of the energy density of the external magnetic field. Stability requirements on the low-frequency magnetohydrodynamic motion of the beams are expected to impose upper limits on the allowed β values.

In the astron system first proposed,[1] the external magnetic mirror field is canceled on axis by the poloidal self-magnetic-field of a beam of relativistic electrons.[4] The use of a beam of high-velocity nonrelativistic ions[2,3] has the important advantage that the incoherent synchrotron radiation is negligible. Field cancelation on axis provides a closed magnetic field suitable for confining a high-pressure plasma, and it requires that the kinetic energy density of the particle beam be a fraction $\beta>1$ of the energy density of the external magnetic mirror field. The magnetohydrodynamic stability of an astron configuration has been previously studied[5] with, however, the restrictive assumption that the relativistic electrons may be treated as rigid and immobile. A detailed analysis of the low-frequency stability of astron- and tokamak-type particle rings including the beam dynamics has recently been completed,[6] and I summarize here some of the results for astron configurations.

The analysis is done in linear cylindrical geometry, with coordinates (r, θ, z), where the z axis coincides with the axis of the unperturbed beam. This assumes that the inverse aspect ratio is small, $a/R \ll 1$, where a is the minor radius of the torus and R is the major radius. The toroidal nature of the actual system enters through the requirement that the perturbation quantities be periodic over the axial length $2\pi R$. Hence the perturbation quantities have a θ, z dependence of the form $\exp(im\theta + ikz)$, with the wave number $k = n/R$, where n and m are integers. Approximations made in describing the beam dynamics restrict consideration to $m=\pm1$, $n\neq0$. Toroidal effects on the beam-particle dynamics are discussed in the final paragraph.

A dense plasma is assumed to fill the region occupied by the particle beam. The plasma may extend beyond the beam, with density $n_p(r)$, out to the radius r_w of the conducting vessel wall. Low-frequency perturbations are considered proportional to $\exp(-i\omega t)$ with $|\omega| \ll \omega_{ci} \equiv |e\vec{B}|/Mc$, where M is the plasma-ion mass, for simplicity $|e|$ (e) is the plasma ion (electron) charge, and $\vec{B}$ is the equilibrium magnetic field. The plasma temperature is assumed sufficiently high so that collisions may be neglected. However, the plasma pressure is neglected, and thus the equilibrium has no plasma currents. For astron-type configurations, the equilibrium magnetic field in the linear geometry is due to the z-direction current density of the particle beam, that is, $\vec{J}_B = J_B(r)\hat{z}$ and $\vec{B} = B_\theta(r)\hat{\theta}$. A possible toroidal ($B_z$) field is not included in the present work.

I consider situations where (a) the Alfvén speed, $v_A \equiv |\vec{B}|/(4\pi n_p M)^{1/2}$, is much smaller than the speed of light c; (b) the plasma frequency, $\omega_p \equiv (4\pi e^2 n_p/m_e)^{1/2}$, is much larger than c/a, where m_e is the electron mass, and a is a measure of the beam radius; and (c) there is local charge neutrality. Under these conditions, the equations of ideal magnetohydrodynamics apply, and we may introduce the plasma Lagrangian displacement $\vec{\xi}$, with $\delta\vec{v}_p = \partial\vec{\xi}/\partial t$, where $\delta\vec{v}_p$ is the plasma velocity perturbation. Thus we find

$$n_p M \frac{\partial^2 \xi}{\partial t^2} + \frac{q n_B}{c}\vec{B} \times \frac{\partial \vec{\xi}}{\partial t} = \vec{\mathfrak{F}} \equiv \vec{F} - \frac{1}{c}\delta\vec{J}_B \times \vec{B}, \tag{1}$$

where $\vec{F}(\vec{\xi}) \equiv (\nabla \times \vec{Q}) \times \vec{B}/4\pi$, $\vec{Q} \equiv \nabla \times (\vec{\xi} \times \vec{B}) = \delta\vec{B}$, $(1/c)\vec{B} \times \partial\vec{\xi}/\partial t = \delta\vec{E}$, $\vec{\xi} \cdot \vec{B} = 0$, $n_B(r)$ is the beam density, and q is the beam-particle charge. Equation (1) is completed by supplying the relation of $\delta\vec{J}_B$ to the perturbation fields $\delta\vec{B}$ and $\delta\vec{E}$.

The operator $\vec{F}(\vec{\xi})$ in Eq. (1) is self-adjoint for the relevant boundary conditions.[7] Thus for a complex eigenfunction solution of Eq. (1) of the form $\vec{\xi}(\vec{x})\exp(-i\omega t)$, we obtain the "energy equation,"

$$\omega^2 T + \omega \mathfrak{L} = W \equiv \frac{1}{8\pi}\int d^3x\, |\vec{Q}|^2 + \frac{1}{2c}\int d^3x\, \vec{\xi}^* \cdot (\delta\vec{J}_B \times \vec{B}), \tag{2}$$

where

$$T \equiv \tfrac{1}{2}M\int d^3x\, n_p |\xi|^2, \quad i\mathfrak{L} \equiv \frac{q}{2c}\int d^3x\, n_B \vec{B} \cdot (\vec{\xi}^* \times \vec{\xi}).$$

T is real and $T \geq 0$, and $\mathfrak{L}$ is real and $\mathfrak{L} \gtrless 0$. The volume integrations extend from $z = 0$ to $2\pi R$ and over the cross section out to the conducting wall. The surface terms involved in obtaining Eq. (2) vanish because of the periodic boundary conditions on z, and the fact that ξ_r vanishes on the conducting wall. For simplicity we consider that the plasma parameters vary smoothly out to the wall so that there are no surface or vacuum terms.

Solving Eq. (2) for ω gives

$$\omega = -\frac{\mathfrak{L}}{2T} \pm \frac{(W + \mathfrak{L}^2/4T)^{1/2}}{T^{1/2}}. \tag{3}$$

Evidently, the beam-plasma perturbations are stable oscillations if W is real and $W + \mathfrak{L}^2/4T \geq 0$. Because $\mathfrak{L}^2/4T \geq 0$, a general sufficient condition for stability is that W be real and positive. Under appropriate conditions, I find that $\vec{\mathfrak{F}}$ in Eq. (1) is a self-adjoint operator on $\vec{\xi}$ so that W is automatically real. Thus there is a variational principle where $\min W > 0$ implies stability. A general, less restrictive, condition for stability, $\min(W + \mathfrak{L}^2/4T) > 0$, is not used in the present work.

We now derive an equation for the center of mass of the beam, $\vec{\epsilon}$, for the kink perturbations ($m = \pm 1$, $n \neq 0$). The beam is considered to be a nonrelativistic ion beam of density $n_B(r)$, and ion mass and charge M_B and q. (For a relativistic electron beam, M_B is replaced by γm_e, where γ is the usual Lorentz factor.) The unperturbed motion of the beam particles in the self-magnetic field $\vec{B}$ is collisionless and consists of a rapid chaotic betatronlike motion within the beam radius. With $\vec{x}_\alpha(t)$ denoting the unperturbed position of the αth beam particle, we write the perturbed position as $\vec{x}_\alpha(t) + \vec{\epsilon}$, where $\vec{\epsilon} = \epsilon_x(z, t)\hat{x} + \epsilon_y(z, t)\hat{y}$ is the same for all particles in the cross section $z = \text{const}$. We consider beams with (a) $|\bar{v}_z| \gg v_A$, where $\bar{v}_z$ is the mean velocity of the beam particles; (b) $\bar{v}_z$ approximately independent of r; (c) $\Delta v_z \ll |\bar{v}_z|$, where Δv_z is the "thermal" spread in the axial velocity of the beam particles; (d) v_x^2, $v_y^2 \ll \bar{v}_z^2$; and (e) $|ka| \gg v_A/|\bar{v}_z| \ll 1$, that is, $n \neq 0$. Under these conditions we find that

$$\epsilon_x = im\left(\frac{q\bar{v}_z}{M_B c}\right)[(k\bar{v}_z)^2 - \omega_\beta^2]^{-1}\langle \delta B_x \rangle, \qquad \epsilon_y = im\epsilon_x, \tag{4}$$

where $\omega_\beta^2 \equiv 2\pi q \bar{v}_z \langle J_B \rangle / M_B c^2$ is the self-field betatron frequency, and $\langle \ldots \rangle$ is an average over the x-y plane weighted with the equilibrium beam current density $J_B(r)$. For long-wavelength perturbations, $|k\bar{v}_z| < \omega_\beta$, the beam displacement is in the direction of the Lorentz force on a beam particle, $q\vec{\bar{v}} \times \delta\vec{B}$. Note that the finite thermal spread Δv_z of an actual beam prevents the resonant factor in Eq. (4) from becoming unbounded. Using the above assumptions and the continuity equation for the beam particles, we may write the beam current-density perturbation as

$$\delta\vec{J}_B = -(\vec{\epsilon}\cdot\nabla)J_B\hat{z} + iJ_Bk\vec{\epsilon}. \quad (5)$$

The terms on the right-hand side of Eq. (2) are written as $W = W_F + W_I$, where $W_F = (8\pi)^{-1}\int d^3x\,|\vec{Q}|^2 \geq 0$, and W_I is the "interaction" term involving $\delta\vec{J}_B$. Using Eqs. (4) and (5) we find

$$W_I = \frac{(\pi R/2c)(q\bar{v}_z/M_Bc)I_B}{(k\bar{v}_z)^2 - \omega_\beta{}^2}\left|\left\langle\frac{1}{r}\frac{\partial}{\partial r}(r\xi_rB_\theta) + ik\xi_zB_\theta\right\rangle\right|^2, \quad (6)$$

where I_B is the total beam current. The numerator of Eq. (6) is ≥ 0. A slight generalization of the derivation of Eq. (6) shows that the right-hand side of Eq. (1) is a self-adjoint operator $\vec{\mathfrak{F}}(\vec{\xi})$.

A sufficient condition for stability of the kink perturbations is $W_I > 0$, or $|kv_z| > \omega_\beta$, for all relevant wave numbers. For $|k\bar{v}_z| > \omega_\beta$, a field perturbation $\delta\vec{B}$ causes a beam displacement which tends to reduce $|\delta\vec{B}|$. Because the smallest $|k|$ allowed is $1/R$, a sufficient condition for stability is $\lambda_\beta > 2\pi R$, where $\lambda_\beta \equiv 2\pi|\bar{v}_z|/\omega_\beta$ is the self-field betatron wavelength. For an astron, the major radius $R = gM_B\bar{v}_zc/qB_e$, where B_e is the external magnetic mirror field, and g is a dimensionless parameter of order unity.[8] (The repulsive effect of the beam self-field acts to increase g, whereas the confining effect of the outer conducting wall acts to decrease g.) Thus, for a constant-density beam ($r \leq a$), the sufficient condition for stability may be written as

$$\beta \equiv 4\pi n_BM_B\bar{v}_z{}^2/B_e{}^2 < 2/g^2. \quad (7)$$

For comparison, the condition for reversal of the direction of the total magnetic field on axis is $\beta \geq (2/\pi g)(R/a)^2$. The stability condition is compatible with field reversal for small aspect ratios, $R/a < (\pi/g)^{1/2}$.

The dynamics of beam particles in a toroidal ring is not fully described by the linear beam model used here. These toroidal effects break the twofold degeneracy of the motion for a given n and are important for the $n=1$ precession mode [ring displacement in the plane of the ring, $\epsilon_x = 0$, $\epsilon_y \propto \exp(iz/R)$, and plasma displacement ξ_r, $\xi_z \propto \sin\theta\exp(iz/R)$]. With toroidal effects included, we find[6] that Eq. (7) is a sufficient condition for low-frequency stability of all kink perturbations but *not* a sufficient condition for stability of the $n=1$ precession mode.

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