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Asymptotic Behavior of the Fixed-Point Hamiltonian and Eigenoperators

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The behavior of the fixed-point Hamiltonian, H^*, and of the eigenoperators is derived for large values of the spatially uniform order parameter. When hyperscaling is valid the limit of H^* is characterized by the exponent, δ, but if hyperscaling is invalid this limit is Gaussian-like. Possible implications of these results, including the significance of marginal operators in approximate calculations, are discussed.

Great progress has been made in recent months in the area of critical phenomena, based on the renormalization-group approach of Wilson.[1,2] In spite of these remarkable developments, however, much still remains to be accomplished, particularly with respect to three-dimensional systems. Exact solutions of exact nonlinear renormalization-group equations[2,3] are of course not likely to be found and progress will therefore be based on approximate calculational schemes, such as have been suggested lately.[4,5] In all such cases one is then faced with the problem of determining the validity of the calculation. At the moment the best guide is to compare the results with those obtained by series expansion techniques, but it is important to know what boundary conditions, if any, must be satisfied by a solution of the renormalization-group equations. In this note we present an initial investigation of this question by determining by means of a heuristic "hydrodynamic" argument the asymptotic behavior of both the fixed-point Hamiltonian, H^*, and the set of eigenoperators $\{O_i\}$ in the limit of large values of the *spatially uniform* order parameter, S. For convenience we limit our attention to the case of a scalar order parameter for which our results are particularly simple, namely, ignoring less important contributions from gradient terms,

$$H^* \sim VCS^{\delta+1} \tag{1}$$

and

$$O_i(\vec{r}) \sim B_i S^{x_i/x_1}, \tag{2}$$

where V is the volume, $x_1 = \frac{1}{2}(d-2+\eta)$, B_i is a constant, and the density $O_i(\vec{r})$ scales like[6,7] r^{-x_i}. The value of C depends on whether or not the hyperscaling relation[8] $2d/(\delta+1) = d-2+\eta$ is valid. If hyperscaling is valid, C is a nonzero constant. Otherwise $C=0$ so that the asymptotic behavior of H^* is Gaussian-like, i.e., H^* approaches a constant which we have taken as zero in (1). Our derivation of (2) assumes hyperscaling, as we discuss below, although the result might be of more general validity. Several interesting results follow from (1) and (2), as we discuss later. One of the more notable ones is that the "boundary conditions" (1) and (2) together with the assumption that one can expand H^* in terms of the basis $\{O_i\}$ require the existence of a marginal operator,[6,7] with $y_i=0$. This suggests that one requirement for an accurate approximate renormalization-group calculation is that one obtains an eigenvalue as close to zero as possible, in order to satisfy (1) as closely as possible. This criterion also has been suggested recently by Bell and Wilson,[9] on the basis of a finite-lattice model calculation. Our result would seem to provide an additional rationale for their suggestion.

We begin by summarizing some facts about the renormalization group. Consider a renormalization-group transformation[2] R_n which when applied to an initial Hamiltonian H generates a sequence of Hamiltonians,

$$H_n(\{S\},\{\mu_i(n)\})=R_n(H(\{M\},\{\mu_i\})),$$

where $\{M\}$ stands for the initial set of order-parameter variables such as the set of Fourier transforms $\{M_{\vec{k}}\}$, $\{S\}$ stands for the new spin variables (such as $\{S_{\vec{k}}\}$), and $\{\mu_i\}$ and $\{\mu_i(n)\}$ are the initial and transformed scaling field variables, respectively. If we follow the prescription of Wilson,[2] i.e.,

$$\exp[-H_{n+1}(\{S_{\vec{k}}\})] = \int_{1<bk<b} D\{S\}\exp[-H_n(\{\zeta S_{bk}\})], \qquad (3)$$

then we have $M_{\vec{k}}=\zeta^n S_{b^n\vec{k}}$ with $\zeta=b^{1-\eta/2}$ and $b>1$, and $F(H_n)=b^{dn}F(H)$, where F is the total free energy. Also, within the linear approximation we have

$$H_n=H^*+\sum_i\mu_i b^{y_in}O_i(\{S\}), \qquad (4)$$

where we have expanded H in terms of the set of eigenoperators O_i conjugate to the scaling fields μ_i, and where $O_i(\{S\})$ is the spatial integral of $O_i(\vec{r})$. The usual length-scaling assumption[6,7] then leads to $y_i=d-x_i$. Having made these preliminary remarks we now discuss the derivation of (1). We apply the renormalization transformation R_n defined in (3) to an initial Hamiltonian which is at criticality in all its field variables $\{\mu_i\}$, except for a nonzero magnetic field, h, so that the correlation length, ξ, is finite. We now observe that for sufficiently large n, H_n differs from H^* only by the renormalized Zeeman term, so that

$$H_n(\{S\},h)=H^*-hV^{1/2}b^{y_1n}S_{\vec{k}=0}. \qquad (5)$$

We now derive (1) from (5) by "hydrodynamic" considerations. We introduce the local variables $M(\vec{r})$ and $S(\vec{r}_n)$ by

$$M(\vec{r})=V^{-1/2}\sum_{\vec{k}}{}' e^{i\vec{k}\cdot\vec{r}}M_{\vec{k}}, \qquad (6)$$

and

$$S(\vec{r}_n)=V_n^{-1/2}\sum_{\vec{k}}\exp(i\vec{k}\cdot\vec{r}_n)S_{\vec{k}}, \qquad (7)$$

where $r_n=r/b^n$, $V_n=V/b^{nd}$, and the sums $\sum_{\vec{k}}'$ and $\sum_{\vec{k}}$ are restricted to $0<k<b^{-n}$ and the first Brillouin zone $(0<k<1)$, respectively. The variable $M(\vec{r})$ is the coarse-grained value of $M_{\vec{k}}$ over a volume of the order of $b^{nd}v_0$, where v_0 is the initial cell volume. Hence, since

$$V^{-1}\sum_{\vec{k}}{}' f(b^n\vec{k})=b^{-nd}V^{-1}\sum_{\vec{k}}f(\vec{k}),$$

we obtain from (6) and (7) the relation[10] $S(\vec{r}_n)=b^{nx_1}M(b\vec{r}_n)$ which maps small values of $M(\vec{r})$ into large values of $S(\vec{r}_n)$. Thus

$$H_n(\{S(\vec{r}_n)\},h)=H_n(\{b^{nx_1}M(\vec{r})\},h),$$

where by construction $M(\vec{r})$ is smoothly varying over a distance $b^n\xi_0$, with $\xi_0=v_0^{1/d}$. Now with h fixed, we choose n sufficiently large such that $b^n\xi_0\gg\xi$ and consider a spatially uniform $M(\vec{r})=M$. Since in this hydrodynamic region the transformation (3) has resulted in integrating out all those fluctuations which contribute to the critical behavior of the system, we can identify $b^{-nd}\times H_n(\{b^{nx_1}M\},h)$ with the total free energy in the sense of Landau,[11] $F(M,h)=VAM^{\delta+1}-VhM$. Here we should note that the $k=0$ component of the order-parameter fluctuation is never eliminated by the renormalization. Thus we have, in terms of S,

$$H_n(\{S\},h)=VAb^{-ndx}S^{\delta+1}-hVb^{y_1n}S, \qquad (8)$$

with $x=(\delta+1)(d-2+\eta)/2d-1$. Now H_n with $h=0$ must be independent of b in the limit of large n, where $H_n\to H^*$, so we immediately have the inequality[12] $x\geq 0$. If $x=0$ (hyperscaling) then (5) and (8) lead to (1), with $C=A\neq 0$. If $x>0$, we find the Gaussian-like asymptotic behavior, i.e., (1) with $C=0$.

Our derivation of (2) proceeds along similar lines. Consider now an initial H which is at criticality except for one of its relevant variables $\mu_i\neq 0$ (so ξ is again finite). Then the renormalized Hamiltonian $H_n(\{S\},\mu_i)$ is given by

$$H_n(\{S\},\mu_i)=H_n^c(\{S\})+\mu_i(n)\int O_i(\vec{r})d\vec{r}, \qquad (9)$$

where H_n^c denotes the renormalized critical Hamiltonian and where $\mu_i(n)=b^{ny_i}\mu_i$ in the linear region. Now assume that the asymptotic behavior of $O_i(\vec{r})$ (which is a functional of $\{S\}$) behaves like $O_i(\vec{r})\sim B_iS^{\theta_i}$ for large values of a spatially uniform spin density. Then the second term of the right-hand side of (9) behaves like $b_i b^{ny_i}\mu_i V S^{\theta_i}$ in the linear region. On the other hand, since the correlation length is finite we have, as before, $F(M,\mu_i)=b^{-nd}H_n(b^{x_1 n}M,\mu_i)$ so that we find[13] from (9), for large S,

$$F(M,\mu_i)=b^{-nd}H_n{}^c(b^{nx_1}M) + B_i b^{n[y_i-d+x_1\theta_i]} V M^{\theta_i}\mu_i. \qquad (10)$$

Hence in order for $F(M,\mu_i)-F(M,0)$ to yield the correct thermodynamic behavior for $\mu_i\neq 0$ ($\mu_i{}^{2-\alpha_i}$ with $\alpha_i=\alpha$ for $\mu_i=T-T_c$, etc.[7]), the last term on the right-hand side of (10) must remain finite as $n\to\infty$. This requires that $\theta_i=x_i/x_1$, i.e., Eq. (2). The same argument can be extended to marginal ($y_i=0$) and irrelevant ($y_i<0$) variables by considering such a $\mu_i\neq 0$ and at the same time maintaining one of the relevant fields different from criticality so as to keep ξ finite.[14] As before, we are led to (2).

We now turn to a brief discussion of our results.

(a) We first note that since our analysis pertains only to the limit of a spatially uniform order parameter, our results are limited to the zero wave-number limit in the functional expansion of the eigenoperators in powers of the Fourier transform of the order parameter. Thus, for example, we correctly predict this zero-momentum limit of the Gaussian-model[2] eigenfunctions, i.e., $O_m\sim S^{x_{m0}/x_{10}}$ with $x_{m0}/x_{10}=m$, using the notation of Ref. 2. However, our analysis does *not* extend to the wave-number–dependent order parameter in the expansion of these operators, such as is associated with the eigenvalues $x_{mp}=m(d-2)/2+p$, with p a positive integer different from zero. We thus do not discuss effects arising from gradients, etc.

(b) These results serve as "boundary conditions" on the renormalization-group equations. It is even possible that one might use (1) to determine δ directly, as it is generally easier to find asymptotic solutions of nonlinear equations than exact (or even approximate) solutions. For example, although analytical solutions of the approximate recursion equations derived by Wilson[2] and Nicoll, Chang, and Stanley[15] have not been determined yet, it is easy to see that the asymptotic solution is $H^*\sim S^{2d/(d-2)}$ (for $2<d\leq 4$) in both cases, so that from (1) we find $\delta=(d+2)/(d-2)$, as is to be expected since $\eta=0$ in both these approximate equations.[16]

(c) We now consider the consequences of assuming that it is valid to expand[7,17] H^* as $\sum\mu_i{}^*O_i$. If this expansion is correct for large S, then (1) and (2) would imply that the limiting behavior of H^* comes from a marginal operator (i.e., $y_i=0$). Hence one criterion for a valid approximate renormalization-group calculation is that there is one eigenvalue very near zero. This might be particularly useful for finite-lattice calculations as has been originally suggested by Bell and Wilson.[9] Although this is a necessary criterion it is of course not always a sufficient one. We also note that the expansion of H^* in terms of O_i together with (2) implies that the coefficient $\mu_i{}^*$ of all the irrelevant eigenoperators are zero.

(d) There is a simple interpretation of the result (2) which follows from considering a spatially varying version of the argument that leads to (2). In this case one would find $O_i(\vec{R})\sim[S(\vec{R})]^{\theta_i}$ (where the spin variables contributing to O_i are assumed to be localized in the vicinity of $\vec{R}$). Then consider the cumulant $\langle O_i(\vec{R})O_i(0)\rangle\equiv G_i(\vec{R})$ which would be proportional to $\langle[S(\vec{R})]^{\theta_i}[S(0)]^{\theta_i}\rangle$. Then assuming hyperscaling, since[18] each power of S contributes a term $x_1=\frac{1}{2}(d-2+\eta)$ to the critical dimension of the cumulant, we would find $G_i\sim R^{-2\theta_i x_1}=R^{-2x_i}$, as is expected since x_i is the critical dimension of O_i. In this regard it is also interesting to speculate that the simplest situation would be (for integral dimension) $O_i=x_i/x_1$, where n_i is some positive integer. If so, this would imply that the critical dimensions of eigenoperators which do not contain gradient terms in any important way are given as simple multiples of a single critical dimension, that of the order parameter. The remaining eigenoperators will then contain gradient terms and their eigenvalues will be $n_i x_1+p$ with some positive integer p. This is in fact the case for the two-dimensional Ising model[19] as well as for the Gaussian model.[2]

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[10]$S(\vec{r}_n)$ is just the renormalized magnetization per original lattice site. The relation between $S(\vec{r}_n)$ and $M(\vec{r})$ is a "micro" version of the well-known relation for the order-parameter correlation function, i.e., $G(\vec{r},\{\mu\}) = b^{-(2-\eta-d)n}G(\vec{r}/b^n,\{R_n(\mu)\})$, where $\{\mu\}$ denotes the set of scaling fields characterizing the Hamiltonian and $\{R_n(\mu)\}$ denotes their renormalized values.

[11]The usual equilibrium free energy is found in the standard way, with $\delta F/\delta M = 0$. For example, this leads to the expected relation $h \sim M^\delta$ at $T = T_c$. More generally, when there is only one nonzero relevant variable, μ_i, we have $M \sim \mu_i^{x_1/(d-x_i)}$.

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[14]The additional terms due to nonzero marginal or irrelevant variables μ_i in the presence of one nonzero relevant field μ_R represent the first-order corrections to scaling exhibiting the weaker confluent singularities proportional to $\mu_i\mu_R d/y_R - \nu y_i$ (see Ref. 7).

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Dual-Frequency Measurement of the Solar Gravitational Microwave Deflection

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The Westerbork synthesis radio telescope was used to measure the solar gravitational deflection of radio waves over a period of 10 days, in October 1973, centered on the occultation of the quasar 3C279 by the sun. Simultaneous measurements at two frequencies (5.0 and 1.4 GHz) allowed the removal of the effects of refraction in the solar corona. The gravitational bending measured was 1.038± 0.033 times that predicted by general relativity. As a by-product of these observations, values for the parameters at a model of the coronal electron density were also obtained.

The solar gravitational deflection of the quasar 3C279 was measured with the Westerbork synthesis radio telescope in October 1973 in a manner similar to that described by Weiler, Ekers, Raimond, and Wellington[1] (hereafter referred to as paper I). For the present observations we added a second observing frequency to virtually eliminate the effects of refraction in the solar corona and hence to avoid the errors arising from the use of coronal electron-density models. However, the component of atmospheric instability which our double-interferometer observing technique could not eliminate was 2 to 3 times larger than it had been in 1972. This then meant that the improvement in accuracy of our results over those of paper I was less than we had hoped. In