Exact Relations among Amplitudes at Critical Points of Marginal Dimensionality

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A renormalization-group analysis of a three-dimensional Ising model with dipolar forces establishes a relation between the specific heat and the correlation length, in the limit $t = (T-T_c)/T_c \to 0$. Specific-heat data are used to predict $\xi \sim 1.41 t^{-1/2}|\ln t|^{1/6}$ Å for $LiTbF_4$. A relation among the magnetization, the specific heat, and the susceptibility is also given. Similar relations are found for n-component short-range models at $d=4$ and $d=4-\epsilon$.

The renormalization-group (RG) analysis of phase transitions[1] predicts that mean-field theory describes the correct critical behavior for dimensionalities $d>d^*$, where $d^*=4$ for short-range interactions[1] and $d^*=3$ for uniaxial dipolar ferromagnets or ferroelectrics[2,3] and for tricritical behavior.[4] At $d=d^*$, the RG equations can be solved exactly, yielding logarithmic corrections to the mean-field behavior.[2,3,5] Thus, experiments at these "marginal" dimensionalities provide a direct check of the RG theory. Indeed, recent measurements of the specific heat per unit volume, C, of the dipolar Ising ferromagnet $LiTbF_4$ confirm the predicted asymptotic behavior, i.e., $C=A|\ln t|^{1/3}$, where $t=(T-T_c)/T_c \to 0$.[6] (We assume $T>T_c$; the case $T<T_c$ will be discussed at the end.) The situation is more complex in measurements of the susceptibility, or of the correlation length, since there the logarithms are superimposed on powers of t. Thus, measurements of the susceptibility and the magnetization of the same material[7] were only able to yield an effective exponent, but not to identify the predicted logarithmic corrections unambiguously.

In this note, motivated by a hypothesis of Stauffer, Ferer, and Wortis[8] termed "two-scale-factor universality," we prove that the amplitudes of the correlation length and of the specific heat are related via a universal constant, which we calculate explicitly and exactly for the dipolar Ising model at $d=d^*=3$ and for the short-range n-component Heisenberg model at $d=d^*=4$. The Stauffer-Ferrer-Wortis hypothesis has previously been proven exactly only for the two-dimensional Ising model[8] and for the spherical model.[9] It has also been checked numerically in several cases.[8] In fact, the hypothesis can be proved generally in the framework of the RG.[10] The universal constant of the n-component theory has been explicitly calculated in this context for $d<4$, to order ϵ^2, where $\epsilon \equiv 4-d$.[11]

Using our result, one can now utilize the very accurate data[6] on the specific heat of $LiTbF_4$ to make predictions for the critical behavior of the correlation length.

The wave-vector-dependent susceptibility of the dipolar Ising model is predicted to behave as[3]

$$\hat{\chi}(T,\vec{q}) = \chi(T)\{1+\xi^2[q^2 - hq_z^2 + g_0(q_z/q)^2]\}^{-1}, \quad (1)$$

where the asymptotic behavior of the "transverse" correlation length ξ is of the form $\xi^2 = \xi_0^2 t^{-1}|\ln t|^{1/3}$. The parameter g_0, which measures the relative strength of the dipolar interaction, is asymptotically temperature independent, and can be estimated by its mean-field value, from measurements well above T_c.[7] The parameter h is weakly decaying to zero as $t\to 0$.

From Eq. (1), one can define a "longitudinal" correlation length, $\xi_{\parallel} = g_0^{1/2}\xi^2$. Both ξ and $\xi_{\parallel}$ can be extracted from the contour in q space of the surface on which $\hat{\chi}(T,\vec{q}) = \frac{1}{2}\chi(T)$.[7]

Our calculation, described below, yields the asymptotic relation

$$\xi^2 \xi_{\parallel} C t^2/k_B = D|\ln t|, \quad D = 3/32\pi, \quad (2)$$

where D is universal and k_B is the Boltzmann constant. Thus, we identify the amplitude of the correlation length as $\xi_0^4 = Dk_B/Ag_0^{1/2}$. For $LiTbF_4$, $A/k_B \approx 5.975\times 10^{-3}$ Å^{-3} (we use a density of 0.022 58 mole cm^{-3} and Ref. 6) and $g_0 \approx 1.56$ Å^{-2},[7] and hence we predict $\xi_0^2 \approx 2.00$ Å^2. Similarly, we predict for the n-component Heisenberg model

$$\xi^d C t^2/k_B = \begin{cases} \dfrac{n(n+8)}{32\pi^2(4-n)}|\ln t|, & d=4, \\[2ex] \dfrac{n(n+8)}{16\pi^2(4-n)\epsilon}[1+O(\epsilon)], & d=4-\epsilon. \end{cases} \quad (3)$$

This result agrees in the limit $n\to\infty$ with that of the spherical model,[9] and relates to the differently defined universal quantity calculated in Ref. 11

via a simple universal ratio. A calculation below T_c[10] gives the additional asymptotic relation[12]

$$t^2 T_c C\chi/M^2 = (4-n)^{-1}, \tag{4}$$

where M is the magnetization, for the three-dimensional dipolar Ising model ($n=1$) and for the n-component short-range model at $d=4$. (Corrections are of order ϵ for $d=4-\epsilon$.)

Our method of calculation is based on a direct integration of the RG recursion relations for the free energy.[13] For the uniaxial dipolar case, the Hamiltonian may be written

$$\mathcal{H} = -\tfrac{1}{2}\int_{\vec{q}}[r_0+q^2-h_0 q_z^2+g_0(q_z/q)^2]\sigma_{\vec{q}}\sigma_{\vec{q}}$$
$$-u_0\int_{\vec{q}}\int_{\vec{q}'}\int_{\vec{q}''}\sigma_{\vec{q}}\sigma_{\vec{q}'}\sigma_{\vec{q}''}\sigma_{-\vec{q}-\vec{q}'-\vec{q}''},$$

with

$$\int_{\vec{q}} \equiv (2\pi)^{-d}\int d^d q, \quad |\vec{q}|<1.$$

In the dipolar regime, $g_0 \gg r_0$. At $d=3$, the RG recursion relations were solved[3] to yield, after l iterations ($l \gg 1$),

$$g_l = g_0 e^{2l}, \quad r_l = r_0 e^{2l}(1+l/l_0)^{-1/3},$$
$$u_l = 2\pi g_0^{1/2} e^l/9(l+l_0), \tag{5}$$

where l_0 is uniquely related to u_0 and g_0. The coefficient h turns out to be irrelevant, and will thus be ignored. The free energy, after l iterations, can be related to the original one via[13]

$$F(r,g,u)/k_B T = \int_0^l G_0(l')e^{-dl'}dl'$$
$$+e^{-dl}F(r_l,g_l,u_l)/k_B T, \tag{6}$$

where, to lowest order in u_l,

$$G_0(l) = \tfrac{1}{2}K_d\langle \ln(r_l+1+g_l\cos^2\theta)\rangle. \tag{7}$$

Here, $K_d^{-1} = 2^{d-1}\pi^{d/2}\Gamma(d/2)$ and the average $\langle\ldots\rangle$ is performed over the angles. We now choose to stop iterations at a value l for which $r_l=1$, and thus $g_l \gg 1$. An explicit calculation at $d=3$ gives the largest singular term in F,

$$F_s/k_B T = -\tfrac{3}{16}\pi K_3 g_0^{-1/2} r_0^2 l_0$$
$$\times[(1+\tfrac{1}{2}l_0^{-1}|\ln r_0|)^{1/3}-1], \tag{8}$$

where we used $l \approx \frac{1}{2}|\ln r_0|$. For small $r_0 \propto t$, this yields $F \propto t^2|\ln t|^{1/3}$. The correlation length changes under iteration as $\xi_l = \xi e^{-l}$.[1] When $r_l=1$, we have $\xi_l = 1+O(l_0/l)$, and hence

$$\xi = e^l = r_0^{-1/2}(1+\tfrac{1}{2}l_0^{-1}|\ln r_0|)^{1/6}. \tag{9}$$

Combining (8), (9), and the definition of $\xi_\parallel$, and taking the limit of small r_0, immediately leads to the result (2). Note that all the nonuniversal parameters u_0, g_0, etc., drop out. The derivation of (3) is similar, and is based more directly on the explicit expressions of Ref. 13.

In practice, one does not perform measurements in the ultimate asymptotic region, in which $1+|\ln r_0|/2l_0$ may be replaced by $|\ln t|/2l_0$. It is therefore wiser to include in the experimental analysis higher-order terms, and fit the measured correlation lengths by a formula like (9), i.e.,

$$\xi^2 = \xi_0^2 t^{-1}|\ln(t/t_0)|^{1/3}. \tag{10}$$

The parameter $\ln t_0$ may result both from a term $\ln[\xi_0^2(2l_0)^{1/3}e^{2l_0}]$, as suggested by (9), as well as from other correction terms, of order $t^{-1}|\ln t|^{-2/3}$ in ξ^2. Indeed, the specific-heat measurements of $LiTbF_4$ were fitted[6] by an analogous expression [similar to (8)], with $t_0 \approx 0.315$.[14] In that case, a recent analysis[15] of higher-order terms in the recursion relations showed that the correction terms are of order $\ln|\ln t|/|\ln t|^{2/3}$, but this is practically indistinguishable from const$/|\ln t|^{2/3}$. Although t_0 is nonuniversal, it is a reasonable estimate to use in (10) the value of t_0 obtained from the specific heat. A comparison of (10) with preliminary measurements[16] indeed leads to an agreement of order 15% in ξ.

To check (4) for $LiTbF_4$, we extracted asymptotic amplitudes for χ and M from Ref. 7, using the formulas

$$\chi = \Gamma_\pm|t|^{-1}|\ln(t_0/|t|)|^{1/3},$$
$$M^2 = B|t||\ln(t_0/|t|)|^{2/3}, \tag{11}$$

and the relation[17] $\Gamma_+/\Gamma_- = 2$. Results are consistent with Eq. (4), but large uncertainties are involved.

Before concluding, we make a few comments. First, Eq. (2) can also be derived from explicit expressions given in Ref. 2. Our results differ from the ones obtained this way by a factor of 2. This can be traced to the factor of $\frac{1}{2}$ in Eq. (7), which seems to be missing in Eq. (28) of Ref. 2. Second, our calculation can easily be repeated for $T<T_c$. Since the amplitudes A and ξ_0^2 are then changed by factors of $4/n$ and $\frac{1}{2}$, respectively,[17] we conclude that our result (2) is unchanged ($n=1$) and that (3) must be divided by n for $T<T_c$. Finally, there are obvious further uses of the idea presented here. For example, various exact relations between amplitudes near tricritical points[4,5] at $d=d^*=3$ can be derived, and used in the complex analysis of experiments.

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Josephson Junctions in Transverse Magnetic Fields

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Recent observations by Rosenstein and Chen of dc Josephson tunnel-current diffraction patterns occurring in a transverse magnetic field are ascribed to surface demagnetizing currents feeding the interior of the junction.

Recently Rosenstein and Chen[1] have reported the observation of a diffraction-pattern modulation of the critical current I_c of a Josephson tunnel junction caused by a magnetic field H applied perpendicular to the plane of the thin films forming the junction. We suggest that this effect is a straightforward consequence of the pattern of surface currents induced on the films by the magnetic field, and that the phenomenological description of Ref. 1 invoking a characteristic "edge penetration length" is not required.

The films, with the geometry shown in the inset of Fig. 1, lie in the x-y plane with $H_\perp$ ($\theta = 90°$) applied in the z direction. Label the upper and lower surfaces of the upper film by 1 and 2 and those of the lower film by 3 and 4. Denote the overlapping portions of surfaces 2 and 3 which form the junction as interior surfaces and the remaining visible portions as exterior surfaces. We suppose that the films are thick compared to the London penetration depth λ but thin compared

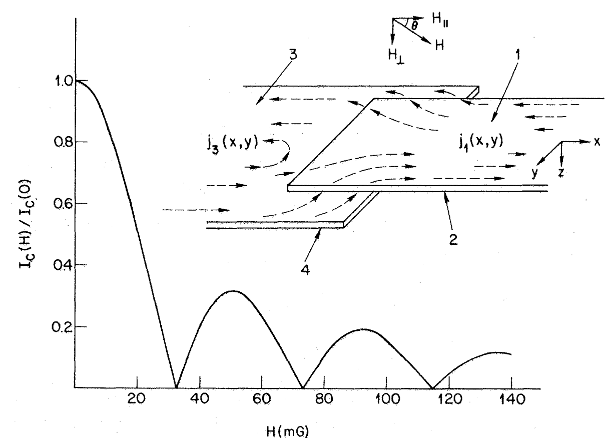

FIG. 1. Calculated diffraction pattern in a transverse field ($\theta = 90°$) for the in-line geometry with equal-width electrodes. The inset shows the actual geometry of the sample used in Ref. 1 which had unequal film widths of 0.74 and 1.00 mm and a common overlay region 0.35 mm in length. Note that the current streamlines appear continuous across the junction boundary, as discussed in the text.