Invariant for a Particle Interacting with an Electrostatic Wave in a Magnetic Field

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A particle moving in a uniform magnetic field is in resonance with an electrostatic plasma wave for an infinite sequence of parallel velocities. Consequently simple perturbation theory breaks down even if the wave amplitude is small. However a technique for calculating invariants in the presence of multiple resonances can be applied and yields results in remarkable agreement with recently published numerical orbit calculations.

In a recent Letter, Smith and Kaufman[1] investigated the interaction between a charged particle and an electrostatic plasma wave in the presence of a uniform magnetostatic field. They remarked that the magnetic field greatly modifies the effect of the interaction with the wave because there is then a *set* of resonant parallel velocities $v_l = (\omega + l\Omega)/k_z$. As a result of these multiple resonances there is no simple invariant as in the case of an unmagnetized plasma. They presented[1] the results of numerical computations of particle orbits for this problem and observed that in some cases an additional constant of motion exists despite the multiple resonances.

The problem discussed by Smith and Kaufman is very similar to that of the motion of a charged particle in a spatially modulated magnetic field.[2] Consequently the method introduced in Ref. 2 for the calculation of invariants in the presence of multiple resonances can easily be applied to the present problem. The invariant calculated in this way is in remarkable agreement with the numerical computations of Smith and Kaufman.

In the wave frame, the Hamiltonian for the interaction of a charged particle with a uniform magnetic field and an electrostatic plasma wave is[1]

$$H(\vec{r},\vec{p}) = (\vec{p} - m\Omega x\hat{y})^2/2m + e\Phi_0 \sin(k_z z + k_\perp x),$$

where $\Omega = eB/mc$ and Φ_0 is the wave amplitude. After the canonical transformations used by Smith and Kaufman this becomes

$$H(z, p_z, \varphi, p_\varphi) = p_z^2/2m + \Omega p_\varphi + e\Phi_0 \sin(k_z z - k_\perp \rho \sin\varphi),$$

where $p_\varphi = mv_\perp^2/2\Omega$ is the canonical angular momentum of gyration, conjugate to the gyrophase φ, and $\rho = (2p_\varphi/m\Omega)^{1/2}$ is the gyroradius. An alternative form, using a Bessel function identity, is

$$H = p_z^2/2m + \Omega p_\varphi + e\Phi_0 \textstyle\sum_l J_l(k_\perp \rho)\sin(k_z z - l\varphi).$$

In the case of wave propagation at 45° to the magnetic field ($k_\perp = k_z$), with introduction of suitable dimensionless variables [$k_\perp = k_z = 1$, $\Omega = 1$, $m = 1$, $\rho = (2P)^{1/2}$], this may be reduced to $H = H_0 + \epsilon H_1$, where

$$H_0 = P + p^2/2,$$
$$H_1 = \textstyle\sum_l J_l(\rho)\sin(z - l\varphi),$$

and we have adopted the notation $p_z \equiv p$, $p_\varphi \equiv P$. (In dimensionless units the small parameter ϵ is just $e\Phi_0$.)

In the absence of interaction with the wave there is a constant of motion P, in addition to the energy, but this is clearly destroyed by the resonant interaction when $\epsilon \neq 0$. To generate a new invariant when ϵ is small but nonzero we note that any constant of motion I satisfies $[I,H] = 0$; then setting $I = I_0 + \epsilon I_1 + \dots$ we obtain a set of recurrence relations

$$[I_{n+1}, H_0] + [I_n, H_1] = 0$$

of which the first two are

$$\partial I_0/\partial\varphi + p\,\partial I_0/\partial z = 0, \tag{1}$$

$$\partial I_1/\partial\varphi + p\,\partial I_1/\partial z + [I_0, H_1] = 0. \tag{2}$$

Equation (1) indicates that I_0 may be an arbitrary function of p. Then, after some elementary analysis similar to Ref. 2, we obtain

$$I_0 + \epsilon I_1 = I_0(p) + \epsilon \frac{dI_0}{dp}\sum_l J_l(\rho)\frac{\sin(z - l\varphi)}{p - l}. \tag{3}$$

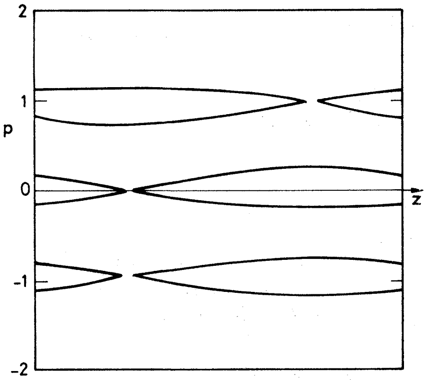

FIG. 1. Surface-of-section plot, $\epsilon=0.025$. Compare Fig. 1(b) of Ref. 1.

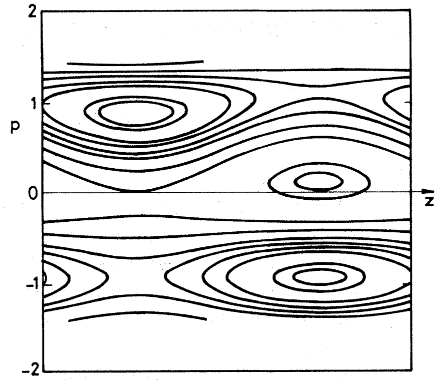

FIG. 2. Surface-of-section plot, $\epsilon=0.1$. Compare Fig. 1(a) of Ref. 1.

The failure of simple perturbation theory near a resonance is apparent, for ϵI_1 is not necessarily small, indeed it is usually infinite, near integral values of p. This breakdown reflects the fact that the topology of the true invariant differs from that of I_0 alone, and I_1 becomes large in an attempt to reduce this, thereby invalidating the perturbation approach. However, this topological change can be accommodated *and* ϵI_1 *can remain small* if we ensure that dI_0/dp vanishes at the resonant points. The arbitrariness in I_0 is sufficient to allow this.

For the present problem, a suitable choice of I_0 is $I_0=\cos(\pi p)/\pi$. Then

$$I=\pi^{-1}\cos(\pi p)-\epsilon\sin(\pi p)\sum_l J_l(\rho)\frac{\sin(z-l\varphi)}{p-l},$$

in which the second term is small when ϵ is small, even at resonances.

A "surface-of-section" plot such as computed by Smith and Kaufman can easily be derived from this invariant and two examples are shown in Figs. 1 and 2. These are calculated for the parameter values used by Smith and Kaufman in their Figs. 1(b) and 1(a), respectively. That is $\varphi=\pi$, $\rho=(2E-p^2)^{1/2}$, and $(2E)^{1/2}=1.48$ with $\epsilon=0.025$ for Fig. 1 and $\epsilon=0.1$ for Fig. 2. The agreement between these figures and those of the numerical orbits calculated in Ref. 1 is quite remarkable.

This agreement confirms that when the behavior of the orbit is nonergodic then, *even in the presence of multiple resonances*, an invariant can be calculated with great accuracy by the method of Ref. 2 when ϵ is small. Of course, neither these calculations, nor so far as we know any other analytic procedure, can determine when the transition from adiabatic to stochastic behavior occurs. The condition quoted by Smith and Kaufman—that the lowest order "islands" overlap[3,4]—is no more than a rough guide. Indeed, our earlier calculations[2] showed that for the particle in a modulated magnetic field the onset of stochastic behavior occurred well before these islands overlapped. Presumably this was due to overlap of much higher-order resonances.[2]

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