PHYSICAL REVIEW LETTERS

VOLUME 35 3 NOVEMBER 1975 NUMBER 18

Nonlinear Evolution Equations—Two and Three Dimensions

M. J. Ablowitz*
Department of Mathematics, Clarkson College of Technology, Potsdam, New York 13676

and

R. Haberman
Department of Mathematics, Rutgers University, New Brunswick, New Jersey 08903

(Received 25 June 1975)

A method is developed which generates a class of nonlinear evolution equations in two and three spatial dimensions from an associated eigenvalue problem and its time dependence. Special cases include the equations describing nonlinear, resonantly interacting, wave envelopes in two and three dimensions; a "nonlinear Schrödinger" equation in two dimensions; and a two-dimensional analog of the Korteweg– de Vries equation.

The inverse-scattering method is used to analyze the initial-value problem for certain nonlinear evolution equations which describe physically important cases of nonlinear dispersive wave propagation. The method was discovered by Gardner, Greene, Kruskal, and Miura.[1,2] Zakharov and Shabat[3] applied the method to the nonlinear Schrödinger equation using ideas presented by Lax.[4] Ablowitz, Kaup, Newell, and Segur developed a technique that generates a class of nonlinear evolution equations that fit into the inverse-scattering method from a general second-order scattering problem.[5,6] Zakharov and Manakov[7] presented a higher-order eigenvalue problem which related to the equations describing a resonantly interacting triad of wave envelopes.

All of the preceding problems have one spatial dimension. However, recently Dryuma[8] obtained a scattering problem corresponding to a two- (spatial) dimensional generalization of the Korteweg–de Vries (KdV) equation. Here, we show that there exist other physically important, multidimensional, nonlinear evolution equations whose corresponding scattering problem is isospectral (i.e., whose spectrum is invariant in time). As special cases we discuss (i) the two- and three-dimensional, nonlinear, partial differential equations describing resonantly interacting wave envelopes; (ii) a two-dimensional "nonlinear Schrödinger" equation; and also (iii) how our procedure yields the result given in Ref. 8. One can now expect the initial-value solution to a class of multidimensional, nonlinear evolution equations to be analyzable. The inverse of the scattering problem presented here must still be studied.

Consider the following matrix scattering problem:

$$\vec{v}_x = i\zeta D\vec{v} + N\vec{v} + B\vec{v}_y. \tag{1}$$

We choose the time dependence of the n-dimensional vector eigenfunction $\vec{v}$ to be

$$\vec{v}_t = Q\vec{v} + C\vec{v}_y + E\vec{v}_{yy}. \tag{2}$$

Here ζ is the eigenvalue and D, N, B, Q, C, and E are $n\times n$ matrices. N_{ij} are the "generalized potentials" and we assume $N_{ii}=0$ and $D_{ij}=d_i\delta_{ij}$. Equations (1) and (2) are the two-dimensional generalization of the eigenvalue problem considered by Ablowitz and Haberman.[9] The form of (1) and (2) was suggested by the work of Dryuma.[8] We remark that scattering problems with higher y derivatives than appear in (1) and (2) will generate other nonlinear evolution equations. In what follows, we shall consider some special cases which result in interesting two-dimensional, nonlinear evolution equations.

The eigenvalues are time invariant if

$$0=[E,B], \tag{3a}$$

$$0=i\zeta[E,D]+[C,B]+[E,N], \tag{3b}$$

$$0=i\zeta[C,D]+[Q,B]+[C,N]+C_x-BC_y+2EN_y, \tag{3c}$$

$$N_t=i\zeta[Q,D]+[Q,N]+Q_x-BQ_y+CN_y+EN_{yy}, \tag{3d}$$

assuming that D, B, and E are constant matrices. Only small modifications of (3a)–(3d) are necessary if B, D, and E are not constant. Equations (3) result from cross-differentiation of (1) and (2) by setting $(\tilde{v}_x)_t=(\tilde{v}_t)_x$ and equating coefficients of y derivatives of $\tilde{v}$. In what follows we shall show, by three examples, how to solve deductively Eqs. (3a)–(3d) and find a nonlinear evolution equation from (3d). Furthermore, by modification of (1) and (2) we can find both higher-order and three-dimensional nonlinear evolution equations.

(i) *Nonlinear wave-envelope interactions (two spatial dimensions).*—Let $E=0$. Equation (3b) becomes $0=[C,B]$, which may be satisfied by $B_{ij}=b_i\delta_{ij}$, $C_{ij}=c_i\delta_{ij}$, where b_i and c_i are taken to be constants. In this case (3c) yields $Q_{ij}=\alpha_{ij}N_{ij}$ $(i\neq j)$, where $\alpha_{ij}=(c_i-c_j)/(b_i-b_j)=\alpha_{ji}$. The diagonal entries of (3d) allow Q_{ii} to be assumed constant, $Q_{ii}=q_i$. Analyzing the off-diagonal entries of (3d) yields a coupled system of nonlinear evolution equations for N_{ij}. In order to remove the eigenvalue ζ present in these equations, we choose $q_i-q_j=i\zeta(d_i-d_j)\alpha_{ij}$; hence

$$N_{ij,t}=\alpha_{ij}N_{ij,x}+\beta_{ij}N_{ij,y}+\sum_{k\neq i,j}(\alpha_{ik}-\alpha_{kj})N_{ik}N_{jk}\quad(i\neq j), \tag{4}$$

where $\beta_{ij}=c_i-a_{ij}b_i$. Equation (4) represents resonant triads ($n=3$) or "multitriads" ($n>3$) if α_{ij} and β_{ij} are real and $N_{jk}=\sigma_{jk}N_{kj}^*$, where $\sigma_{ik}\sigma_{kj}=-\sigma_{ij}$ for all $i>k>j$ (see also Ref. 9). For example, when $n=3$, (4) becomes the two- (spatial) dimensional three-wave interaction equations. The coefficients α_{ij} and β_{ij} are the x and y linear group velocities of the wave envelopes.[10] Apart from degenerate cases, they may be prescribed arbitrarily. If all σ_{ij} are negative, then this corresponds to a stable interaction. Otherwise we have explosive instability.[7]

(ii) *A two-dimensional nonlinear Schrödinger equation.*—Another solution of (3) corresponds to $E_{ij}=e_i\delta_{ij}$, $B_{ij}=b_i\delta_{ij}$, where b_i and e_i are assumed to be constants. Proceeding as before, a systematic solution of (3a)–(3c) shows $C_{ij}=\gamma_{ij}N_{ij}$ $(i\neq j)$, where $\gamma_{ij}=(e_i-e_j)/(b_i-b_j)=\gamma_{ji}$, and C_{ii} are constants. Further

$$Q_{ij}=\frac{C_{ii}-C_{ii}}{b_i-b_j}N_{ij}-i\zeta\frac{d_i-d_i}{b_i-b_j}\gamma_{ij}N_{ij}+\frac{\gamma_{ij}}{b_i-b_j}N_{ij,x}+\frac{2e_i-b_i\gamma_{ij}}{b_i-b_j}N_{ij,y}$$
$$+\frac{1}{b_i-b_j}\sum_{k\neq i,j}(\gamma_{ik}-\gamma_{kj})N_{ik}N_{kj}\quad(i\neq j), \tag{5}$$

and

$$Q_{ii,x}-b_iQ_{ii,y}=\sum_{k\neq i}\left[\frac{\gamma_{ik}}{b_k-b_i}(N_{ik}N_{ki})_x+\frac{2e_i-b_i\gamma_{ik}}{b_k-b_i}(N_{ik}N_{ki})_y\right]. \tag{6}$$

By substitution into the off-diagonal entries of (3d), a system of equations for N_{ij} results. When $n=2$, the eigenvalue ζ is removed by requiring that $C_{ii}=2i\zeta e_i(d_i-d_j)/(b_j-b_j)$ and

$$\lim_{|x|,|y|\to\infty}(Q_{ii}-Q_{jj})=-\zeta^2\gamma_{ij}(d_i-d_j)^2/(b_i-b_j)\quad(i\neq j).$$

In this manner, we generate (for $i \neq j$, $n=2$),

$$N_{ij,t} = \frac{e_i - e_j}{(b_i - b_j)^2} N_{ij,xx} + 2\frac{b_i e_j - e_i b_j}{(b_i - b_j)^2} N_{ij,xy} + \frac{e_i b_j^2 - b_i^2 e_j}{(b_i - b_j)^2} N_{ij,yy} + (\overline{Q}_{ii} - \overline{Q}_{jj}) N_{ij}, \tag{7}$$

where $\overline{Q}_{kk}$ is the difference between Q_{kk} and its asymptotic limit at infinity,

$$\overline{Q}_{kk} = Q_{kk} - \lim_{|x|,|y| \to \infty} Q_{kk}.$$

Hence $\overline{Q}_{kk}$ vanishes as $|x|, |y| \to \infty$. Letting $N_{21} = \pm N_{12}^*$, b_i be real, and e_i be purely imaginary, we find a single consistent nonlinear Schrödinger equation in two dimensions. Writing $N_{12} = A$, (7) becomes

$$i\,\partial A/\partial t = D_{\bar{x}}^2 A + i\varphi A, \tag{8}$$

where $\varphi \equiv \overline{Q}_{11} - \overline{Q}_{22}$, and

$$D_{\bar{x}}^2 \equiv \frac{i}{(b_1 - b_2)^2}\left[(e_1 - e_2)\frac{\partial^2}{\partial x^2} + 2(b_1 e_2 - e_1 b_2)\frac{\partial^2}{\partial x\,\partial y} + (e_1 b_2^2 - b_1^2 e_2)\frac{\partial^2}{\partial y^2}\right], \tag{9a}$$

$$\overline{Q}_{ii,x} - b_i \overline{Q}_{ii,y} = \pm\left[\frac{\gamma_{ij}}{b_j - b_i}(AA^*)_x + \frac{2e_i - b_i\gamma_{ij}}{b_j - b_i}(AA^*)_y\right] \quad (i \neq j,\ i = 1, 2). \tag{9b}$$

The two-dimensional operator $D_{\bar{x}}^2$ is hyperbolic if $e_1 e_2 > 0$, and elliptic if $e_1 e_2 < 0$. Equations (8) and (9b) form a system of nonlinear partial differential equations. If the amplitude A is one-dimensional (in any direction), then (9b) can be integrated and the system reduces to the standard nonlinear Schrödinger equation with cubic nonlinearity.

(iii) *A two-dimensional KdV equation.*—Equations (3) in the 2×2 case also can yield the equation discussed by Dryuma.[8] The systematic solution of (3) is initiated by letting $B_{11} = B_{12} = B_{22} = 0$, B_{21} being assumed constant. Ignoring identity solutions to commutator equations, (3a) yields $E = kB$, where we assume k is constant, and (3b) yields $C = kN + i\zeta kD + \alpha\beta$, where α is *not* constant. (3c) is solved by $Q = \alpha N + i\zeta\alpha D + \nu B + G$, where $G_{ij} = g_i \delta_{ij}$, $g_1 - g_2 = \alpha_x + kN_{21,x}/\beta_{21}$, $N_{12} = \text{const}$. Equation (3d) now yields equations for the potential N_{21}. After some manipulation, including the elimination of the ζ terms, we find $d_2 = -d_1$, as well as the functions α, ν, g_1, and g_2, and the evolution equation found by Dryuma,[8]

$$N_{21,t} = (k/4\beta_{21}N_{12})(-6N_{12}N_{21}N_{21,x} + N_{21,xxx} + 3\beta_{21}^2 N_{12}^2 \int^x N_{21,yy}\,dx'). \tag{10}$$

Equation (10) is a two-dimensional equation which reduces to the KdV equation if the problem is independent of y. Recently, special solutions to (10) have been found by Chen[11] using Bäcklund transformations.

(iv) *Three-dimensional, nonlinear evolution equations.*—To generate three-dimensional, nonlinear evolution equations, (1) and (2) are generalized to

$$\tilde{\mathbf{v}}_x = i\zeta D\tilde{\mathbf{v}} + N\tilde{\mathbf{v}} + B\tilde{\mathbf{v}}_y + G\tilde{\mathbf{v}}_z, \tag{11}$$

$$\tilde{\mathbf{v}}_t = Q\tilde{\mathbf{v}} + C\tilde{\mathbf{v}}_y + F\tilde{\mathbf{v}}_z + E\tilde{\mathbf{v}}_{yy} + H\tilde{\mathbf{v}}_{yz} + J\tilde{\mathbf{v}}_{zz}. \tag{12}$$

Again D, N, B, G, Q, C, F, E, H, and J are $n\times n$ matrices. Equations similar to (3) can be obtained. However, to be brief, we will discuss only the example of triad resonance which follows from taking $E = H = J = 0$ and $B_{ij} = b_i\delta_{ij}$, $G_{ij} = g_i\delta_{ij}$, $C_{ij} = c_i\delta_{ij}$, and $F_{ij} = f_i\delta_{ij}$. The eigenvalues are time invariant if $Q_{ij} = \alpha_{ij}N_{ij}$ $(i \neq j)$, where $\alpha_{ij} = (c_i - c_j)/(b_i - b_j) = (f_i - f_j)/(g_i - g_j) = \alpha_{ji}$. We find that $Q_{ii} - Q_{jj} = i\zeta(d_i - d_j)\alpha_{ij}$ eliminates ζ from the evolution equation; hence we find the three-dimensional version of (4),

$$N_{ij,t} = \alpha_{ij}N_{ij,x} + \beta_{ij}N_{ij,y} + \epsilon_{ij}N_{ij,z} + \sum_{k\neq i,j}(\alpha_{ik} - \alpha_{kj})N_{ik}N_{kj}, \tag{13}$$

where $\beta_{ij} = c_i - \alpha_{ij}b_i$ and $\epsilon_{ij} = f_i - g_i\alpha_{ij}$.

The above equations are the result of a general method which generates multidimensional, nonlinear evolution equations appropriate for an inverse-scattering analysis. No assumptions on the ζ dependence of Q are made. This contrasts with the one-dimensional problems[5,6] in which a different evolution equation exists corresponding to each assumption on Q. To achieve the generality of the one-dimensional case, it is only necessary to modify the y dependence of (1) and (2) [and z dependence in

(11) and (12)]. Notice that when $E=0$ in (2) the evolution equation (4) contains one derivative in y, whereas if $E\neq 0$ two derivatives in y appear in the evolution equation [see (7) and (10)]. Higher derivatives work in a similar way. This is analogous to the role of the dispersion relation in the one-dimensional problem.[6]

Note added.—We have become aware that Zakharov and Shabat[12] have obtained, by a different approach, the two- (spatial) dimensional three-wave interaction equations and Dryuma's result in Ref. 8.

*Work supported by National Science Foundation Grants No. GP32829X and No. GA27727A. Alfred P. Sloan Foundation Research Fellow, 1975–1977.

[1]C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura, Phys. Rev. Lett. 19, 1095 (1967).

[2]C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura, Comments Pure Appl. Math. 27, 97 (1974).

[3]V. E. Zakharov and A. B. Shabat, Zh. Eksp. Teor. Fiz. 61, 118 (1972) [Sov. Phys. JETP 34, 62 (1972)].

[4]P. D. Lax, Comments Pure Appl. Math. 21, 467 (1968).

[5]M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur, Phys. Rev. Lett. 31, 125 (1973).

[6]M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur, Stud. Appl. Math. 53, 249 (1974).

[7]V. E. Zakharov and S. V. Manakov, Pis'ma Zh. Eksp. Teor. Fiz. 18, 413 (1973) [JETP Lett. 18, 243 (1973)].

[8]V. S. Dryuma, Pis'ma Zh. Eksp. Teor. Fiz. 19, 753 (1974) [JETP Lett. 19, 387 (1974)].

[9]M. J. Ablowitz and R. Haberman, to be published.

[10]D. J. Benney and A. C. Newell, J. Math. Phys. 46, 133 (1967).

[11]H. Chen, to be published.

[12]V. E. Zakharov and A. B. Shabat, Funkts. Anal. Prilozh. 8, No. 3, 43 (1974) [Func. Anal. Applic. 8, 226 (1974)].

Theories of Gravity with Structure-Dependent γ's

M. Camenzind

Institute for Theoretical Physics, University of Berne, 3012 Berne, Switzerland

(Received 28 July 1975)

A gauge theory for the Lorentz connection of a space-time describes geometries of stars with variable parametrized post-Newtonian parameter γ; its value depends explicitly on the structure of a star: The value of γ for a neutron star is different from that for the solar system ($|\gamma_{\odot}-1|\leq 10^{-2}$). Because of the absence of a "local parity invariance" γ is positive for ordinary stars and γ is negative for sources of gravity consisting of antimatter.

Several recent Letters[1-3] present various aspects of two-parameter static and spherically symmetric space-time geometries, generated by Yang's gravitational field equations $R_{ab;c}-R_{ac;b}=0$.[4] The geometric background of these field equations has partially been discussed by Kilmister and Newman,[5] they have been used by Lichnerowicz[6] in his quantization of the gravitational field, and they appear in Bel's investigation of the super-energy-momentum tensor of the gravitational field[7]; the Lagrangian for Yang's equations has been noted by Eddington[8] as an alternate choice for gravitation. The space-time connections discussed in Refs. 1–3 may be classified by means of the family of vacuum solutions of the curvature dynamical equations,[9,10] which are the natural gauge equations of second order for the connection of the Lorentz-frame bundle of a static and spherically symmetric space-time (here given in terms of Schwarzschild's coordinates, $g_{00}=e^{2\mu}$, $g_{11}=e^{-2\lambda}\equiv\Delta^{-2}$),

$$r^{-2}(r^2f')'-2r^{-2}f=-e^{-1\lambda}f'\chi(\Delta',f)+\kappa j^{(1)0}e^{2(\mu-\lambda)}, \qquad (1)$$

$$\tfrac{1}{2}\Delta^{2\prime\prime}-r^{-2}(\Delta^2-1)=-e^{-\mu}f\chi(\Delta',f)+\kappa e^{-\lambda}r^2S^{(3)2}, \qquad (2)$$

$$\chi(\Delta',f)\equiv\Delta'-e^{-\mu}f. \qquad (3)$$

$f=(e^{\mu})'\Delta$ represents the gravitational force measured in the static frame of reference; $j^{(1)0}(r)$ and $S^{(3)2}(r)$ are the only nonvanishing components of the external Lorentz current $\mathcal{J}^Q$ which couples matter to geometry. Any static and spherically symmetric connection which is regular in the