Nonlinear Optical Standing Waves in Overdense Plasmas

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A new class of exact standing-wave solutions of the Maxwell cold-plasma equations with fixed ions is described. The waves are transverse, and circularly polarized. These solutions permit an analytical treatment of intense wave penetration into inhomogeneous plasmas with fixed ions.

Analytical investigations of the propagation of very intense electromagnetic radiation in overdense plasmas ($\omega_p > \omega$) customarily begin with the "exact" traveling-wave solutions of Akhiezer and Polovin.[1-3] These solutions, which satisfy Maxwell's equations and the relativistic electron equation of motion for fixed ions, are suitable only for the treatment of moderately inhomogeneous plasmas in which reflection is negligible.[3] In the extreme case of total reflection, a normally incident wave in a stratified medium must resemble more a standing wave than a traveling wave. We have found that exact *standing*-wave solutions, similar to the Akhiezer-Polovin solutions, exist in both homogeneous and inhomogeneous cold plasmas, and that they may be written down in closed form for the homogeneous case. These solutions lead to analytical expressions for the reflection point of circularly polarized standing waves in plasmas of arbitrary (monotonically increasing) ion density variation. While extremely interesting for the insight they give into the reflection of intense waves, the solutions imply a redistribution of electron density which must exert great forces on the ions, rendering the approximation of immobile ions unrealistic. Nevertheless, the maximum penetration of a pulse into a plasma boundary depends upon the electron response during the initial formation of the standing wave, before the ions can move far. Thus if one must choose a *steady-state* solution which leads to a plausible criterion for the penetration depth of reflected waves, the solution with fixed ions is preferable. The steady-state solution for standing waves with movable ions leads to bunching of the plasma at the nodes of the electric field.[4,5] At zero temperature, there is no restriction on the field strength or penetration depth in this case. A good picture of plasma bunching in nearly standing waves of linear polarization has been obtained numerically by Valeo and Estabrook for warm plasmas.[4] The field strengths employed in their work are at least an order of magnitude less than those for which the effects considered here are important. Nevertheless their conclusion that the resulting thin sheets of plasma are unstable against transverse bubble-type instabilities at finite temperature will certainly apply here as well.

Our starting equations are those of Akhiezer and Polovin,

$$\nabla\times\vec{B} = (4\pi/c)ne\vec{v} + (1/c)\partial\vec{E}/\partial t, \quad \nabla\cdot\vec{B} = 0,$$
$$\nabla\cdot\vec{E} = 4\pi e(n-n_0), \quad \nabla\times\vec{E} = -(1/c)\partial\vec{B}/\partial t,$$
$$\partial\vec{p}/\partial t + (\vec{v}\cdot\nabla)\vec{p} = e\vec{E} + (e/c)(\vec{v}\times\vec{B}), \tag{1}$$

where $\vec{p} = m\vec{v}/(1-v^2/c^2)^{1/2}$, and the independent variables are z and t. The ion density $n_0(z)$ is assumed stationary. In a standing wave, the components of the fields and electron density are required to have the factored form $E_i(z,t) = F_i(z)f_i(t)$, etc. This very strong condition has the following consequences for inhomogeneous plasmas: (1) The wave must be circularly polarized. (2) The longitudinal momentum p_z must vanish. The detailed proof of these statements will be reported elsewhere. The important conclusion is that the transverse components of $\vec{E}$ and $\vec{B}$ may be derived from a vector potential $\vec{A}$ of the form

$$e\vec{A}/mc^2 = F(\omega z/c)(-\hat{x}\cos\omega t + \hat{y}\sin\omega t), \tag{2}$$

with $\vec{p} = -e\vec{A}/c$. The longitudinal field is given by a time-independent scalar potential $\Phi(z) = (mc^2/e)\varphi(\omega z/c)$. These expressions when inserted into Eqs. (1) yield

$$\varphi' = -d(1+F^2)^{1/2}/d\zeta, \tag{3}$$
$$-\varphi'' = (\omega_p^2 - \omega_{p0}^2)/\omega^2, \tag{4}$$
$$F'' + F = (\omega_p^2/\omega^2)F/(1+F^2)^{1/2}, \tag{5}$$

where $\zeta = \omega z/c$, $\omega_p^2 = 4\pi ne^2/m$, $\omega_{p0}^2 = 4\pi n_0 e^2/m$, and primes denote $d/d\zeta$.

Eliminating φ and ω_p/ω from (5), one finds an equation for F with the following properties: F may be regarded as the displacement of a classical particle whose variation with "time" ζ is described by the Lagrangian

$$L=\tfrac{1}{2}F'^2/(1+F^2)-\tfrac{1}{2}F^2+\alpha(1+F^2)^{1/2} \tag{6}$$

where $\alpha=\omega_{p0}^2/\omega^2$ may be a function of ζ. This Lagrangian may be derived from the Lagrangian density for the zero-temperature relativistic-electron fluid plus electromagnetic fields and fixed ions

$$\mathcal{L}=(E^2-B^2)/8\pi + n[-mc^2(1-v^2/c^2)^{1/2}+e\vec{v}\cdot\vec{A}/c]+(n_0-n)e\Phi$$

by substituting the appropriate expressions for $\vec{A}$, Φ, and $\vec{p}$, and using (3)–(5). The associated Hamiltonian $H=P^2/2M(F)+V(F)$ [with $M(F)=(1+F^2)^{-1}$, $V(F)=\frac{1}{2}F^2-\alpha(1+F^2)^{1/2}$, $P=M(F)F'$] is a constant of the motion when the ion density is homogeneous.

Equations (3) and (4) show that the electron density can vanish when

$$d^2(1+F^2)^{1/2}/d\zeta^2<-\alpha. \tag{7}$$

Where there are no electrons, the Lagrangian reduces to its vacuum form for standing waves, $2L_0=F'^2-F^2$. The electron density function must be obtained from (3) and (4) and the condition of charge neutrality $\int(\alpha_e-\alpha)\,d\zeta=0$, where $\alpha_e=\omega_p^2/\omega^2$. The integral here need only cover a spatial quarter period ζ_p in the homogeneous case. In this case, the value F_d of the field at the boundary between depleted and nondepleted regions may be found by combining the charge-neutrality condition with the boundary condition which matches the logarithmic derivatives of the solution in the two regions at the depletion boundary. The resulting implicit equation may be written

$$G^{-1}\tan G=\alpha(1+F_d^{\,2})^{1/2}/F_d^{\,2}, \tag{8}$$

where $G^2=2(F_d/\alpha)^2[H-V(F_d)]$. Once F_d is found the depletion boundary may be determined from the explicit form of the solution in the undepleted region, given below.

For homogeneous ion density, the potential $V(F)$ has a single minimum for $\alpha<1$, and a double well for $\alpha>1$ (the overdense case). Figure 1 shows plots of F' versus F (or B versus transverse E) for $\alpha=\frac{4}{3}$, which imply three kinds of motion for F when $\alpha>1$: (1) oscillations about zero when the maximum field F_m exceeds $F_T\equiv 2(\alpha^2-\alpha)^{1/2}$; (2) oscillations about a bias field $F_B=(\alpha^2-1)^{1/2}$ when $F_m<F_T$; (3) motion with infinite period along the separatrix $F_m=F_T$. These formulas are easy to obtain from the expression for H. It is also easy to find from (7) the greatest maximum field $F_m=F_d$ for which no electron depletion occurs. As F approaches F_d from below, α_e given by (4) must approach zero. This gives

$$F_d=\{(\alpha/2)[\alpha+(4+\alpha^2)^{1/2}]\}^{1/2}. \tag{9}$$

Figure 2 shows F_d and F_T versus α. Also shown

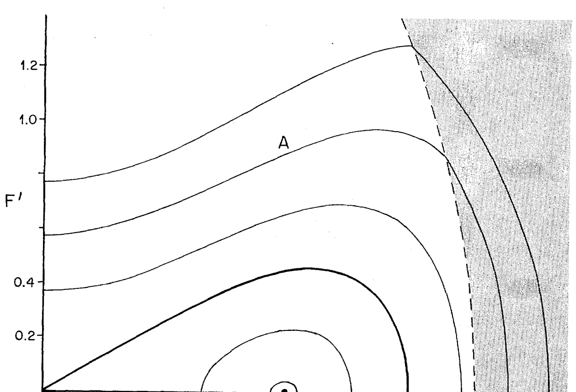

FIG. 1. Fields B (F') versus transverse E (F) in units $m\omega c/e$ for a quarter period when $\omega_{p0}^2/\omega^2=\frac{4}{3}$. The depletion region where there are no electrons is shaded. Curve A corresponds to Fig. 3.

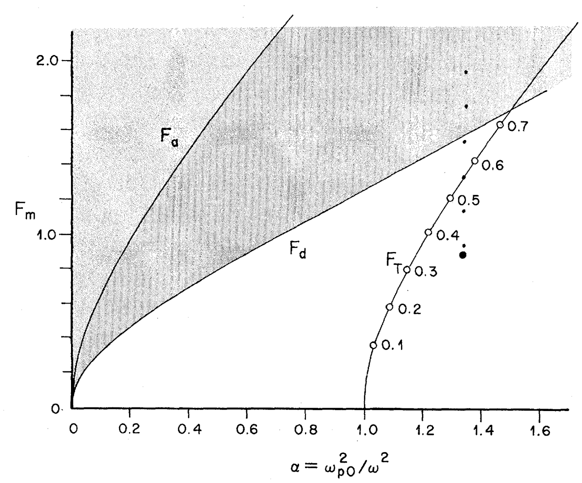

FIG. 2. Maximum field amplitude F_m characterizing solutions in nondepleted regions versus ion density in units $m\omega^2/4\pi e^2$. Dots correspond to curves in Fig. 1. Open circles are fields at turning point corresponding to vacuum field strength F_0 indicated from Eq. (13). Region where depletion occurs is shaded.

is the curve $F_m = F_a \equiv 2(\alpha^2+\alpha)^{1/2}$, for which the solution $F(\zeta)$ has a simple form given below. Because the solutions are awkward to work with when depletion occurs, we shall limit our attention to the region $F_m \leq F_d$. This implies $F_m \leq \sqrt{3}$ if we omit consideration of the biased oscillations of case (2) above. For 1-μm radiation, a case of experimental interest, this limit implies peak fields *in the plasma* less than $\sqrt{3}\,m\omega c/e \approx 1.86\times10^8$ esu, or power densities $\approx 4\times10^{18}$ W/cm^2.

The equations for F may be integrated explicitly in terms of Jacobian elliptic functions for all field strengths. When $F_T \leq F_m \leq F_a$, the quarter period is $\zeta_p = \gamma^{-1}K(m)$, where K is the complete elliptic integral of the first kind, and the field amplitude is

$$F(\zeta) = 2F_m \operatorname{cn}(\gamma\zeta)/[2+(\epsilon_m - 1)\operatorname{sn}^2(\gamma\zeta)] \tag{10}$$

where $\epsilon_m^{\,2} = 1 + F_m^{\,2}$, $\gamma^2 = \epsilon_m - \alpha$. The parameter m of the elliptic functions is $m = (\epsilon_m - 1)(2\alpha + 1 - \epsilon_m)/4(\epsilon_m - \alpha)$. When $F_m = F_a$, $m = 0$,

$$F(\zeta) = F_a \cos(\gamma'\zeta)/[1+\alpha\sin^2(\gamma'\zeta)], \tag{11}$$

where $\gamma'^2 = \alpha + 1$. For the important nonperiodic limit $F_m = F_T$, $m = 1$,

$$F(\zeta) = F_T \cosh(\gamma''\zeta)/[1+\alpha\sinh^2(\gamma''\zeta)], \tag{12}$$

where $\gamma''^2 = \alpha - 1$. The formulas for $F_a \leq F_m$ are somewhat more complicated and will be given elsewhere. These solutions are necessary for the evaluation of the boundaries of the depletion region from Eq. (8), indicated on Fig. 1 for the case $\alpha = \frac{4}{3}$. The F' versus F trajectories in this region are segments of ellipses. Figure 3 shows F and the electron density α_e for trajectory A of Fig. 1.

The most interesting application of these solutions is in estimating the properties of standing waves in inhomogeneous plasmas. For this purpose, it is useful to have the integral invariant $I = \oint P\,dF$ which is nearly independent of ζ when $\alpha(\zeta)$, the ion density, is not constant. The phase integral may include contributions from both the depleted and undepleted regions, and the appropriate canonical momenta must be employed for each region. Although the canonical momentum is discontinuous across the depletion boundary, I is still a continuous function of F_m and α, and remains an invariant. The usual arguments[6] for invariance of the phase integral can be applied to the sum of contributions from the depleted and undepleted regions. I can be evaluated in terms of elliptic integrals for all field strengths, but the case $F_m = F_T$ is particularly important, since this is the turning point in the sense that the spatial quarter period ζ_p becomes infinite there, and unbiased oscillations are forbidden for smaller fields or higher densities. To find an explicit formula for the vacuum field strength F_0 required for a turning point at $\alpha(\zeta)$, we set the adiabatic invariant at the turning point I_T equal to the invariant I_0 for vacuum propagation with amplitude F_0:

$$I_0 = \pi F_0^{\,2} = I_T = 8\alpha\theta - 8(\alpha-1)^{1/2}, \tag{13}$$

where $\sin^2\theta = (\alpha - 1)/\alpha$. This procedure accounts well for the increasing amplitude of the wave as it approaches the turning point, and is similar to

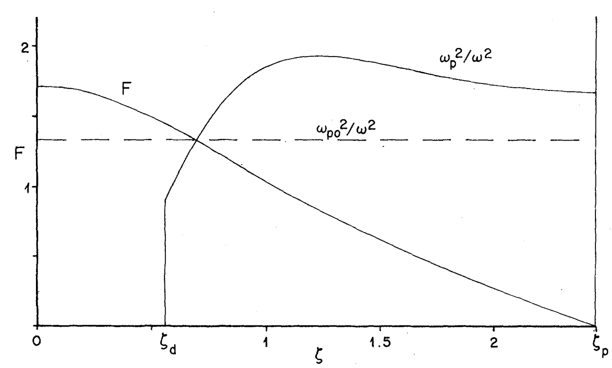

FIG. 3. F: transverse electric field versus $\zeta = \omega z/c$ over a quarter period. This corresponds to curve A in Fig. 1. $\omega_p^{\,2}/\omega^2$: electron density. $\omega_{p0}^{\,2}/\omega^2$: ion density. The areas under these two curves are equal to satisfy charge neutrality.

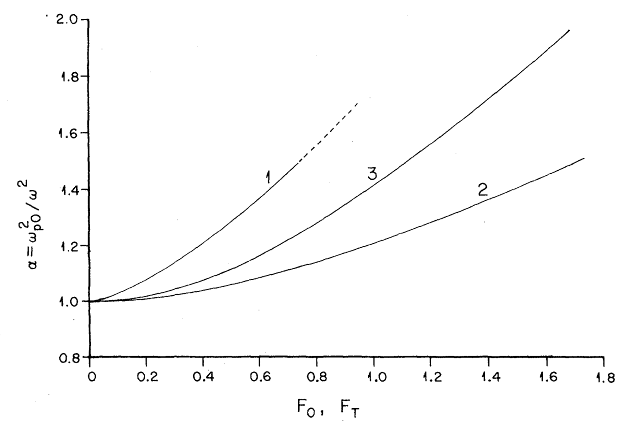

FIG. 4. Ion density at turning point versus (1) incident vacuum field F_0 from Eq. (13); (2) field in medium F_T; (3) least maximum field in medium for circularly polarized traveling waves, from Ref. 1. The dashed line indicates depletion near turning point, where Eq. (13) is incorrect.

the WKB approximation for linear waves. In contrast with the linear case, the amplitude estimated this way remains finite at the turning point, and is indicated on Fig. 2 for a few incident fields.

Figure 4 shows the ion density at the turning point as a function of *incident* field strength, from (13). The corresponding ion density versus the maximum field *at the turning point* $F_T = 2(\alpha^2 - \alpha)^{1/2}$ is also shown. This curve may be compared with the condition of Akhiezer and Polovin[1,2] for the existence of circularly polarized traveling waves in a uniform overdense medium, $F \geq (\alpha^2 - 1)^{1/2}$. A more interesting condition, $F_0 \gtrsim (\alpha/Z)^{1/2}$ derived by Max and Perkins[3] for propagation of very strong waves in a nonuniform medium, depends on the scale length of the plasma $Z = (d\ln\alpha/d\xi)^{-1} \equiv \omega L/c$, and could therefore be plotted on Fig. 4 only for a definite choice of scale length. In any case, this condition applies to fields $F_0 \gg 1$ for which (13) must be altered because of depletion effects near the turning point.

At the field strengths we have been considering, thermal effects are negligible unless the electron thermal velocity v_t approaches the speed of light. A value of v_t typically employed in laser-induced-plasma simulations[4] is $c/10$. Much more serious than neglecting the temperature is our assumption of fixed ions. The ion density is known to bunch at the nodes of the field after the standing wave is formed. The extent to which the transient evolution of these striations alters our predictions of penetration depth cannot be investigated in the context of our solutions.

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Intensity and Linewidth of Rayleigh Scattering near the Double Plait Point of the System Ne-Kr†

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From the intensity and Rayleigh linewidth of light scattered by the system Ne-Kr near the temperature minimum of the critical line (double plait point), we have verified the phenomenological rule that, for constant overall composition, the critical exponents are independent of the path of approach to the critical line, except along a path which is asymptotically parallel to the critical line where they assume twice their usual values.

Phase diagrams of binary systems show a variety of behavior.[1] Among them, fluid-fluid equilibria of the second type are of special interest: Their critical line passes through a temperature minimum as a function of pressure (or composition) at the so-called "double plait point."

As discussed by Griffiths and Wheeler,[2] the critical divergence for a given quantity must be independent of the path of approach to a point of the critical line at constant overall composition except when approaching the critical line tangentially. In this latter case, it is predicted that the critical exponents will differ from their conventional values. Figure 1 shows several possible paths of approach to the critical line in the P-T plane.

The purpose of this experimental investigation was (1) to verify that the critical exponents for a given quantity, in a region where P_c varies smoothly with temperature, should be the same whether the variable is taken to be $P - P_c$ or $T - T_c$, the composition x_c being held constant (this prediction follows from the fact that in either case the path followed is parallel to the coexistence surface in the "field" representation introduced by Griffiths and Wheeler[2]); and (2) to