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Uniqueness of the Kerr Black Hole

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The family of Kerr solutions, with $|a|<m$, is shown to be the unique pseudostationary family of black-hole solutions of the Einstein vacuum field equations when the event horizon is assumed to be nondegenerate.

According to the theorem of Israel[1,2] the static geometry of the vacuum space-time exterior to a nonrotating, isolated, uncharged black hole is determined by the Schwarzschild solution of Einstein's vacuum field equations. The possible equilibrium states of a rotating, uncharged, isolated, black hole have, until now, not been so completely determined. However, it has been shown by Hawking[3] and Hawking and Ellis[4] that the geometry of the (pseudo-) stationary space-time manifold exterior to the event horizon of a rotating black hole must be axisymmetric and the topology of the event horizon must be $S^2\otimes R^1$. Furthermore, Carter[5,6] has shown, in the latter case, that when the event horizon is assumed to be nondegenerate the possible vacuum solutions form discrete continuous families each depending on, at most, two parameters. It is the purpose of this paper to demonstrate that in fact only one such family exists; namely, the Kerr family with, in the usual notation, $|a|<m$.

Carter[6] has shown that the geometry external to the boundary of an axisymmetric, pseudostationary black hole may be determined by solving two equations for two quantities X and Y which are scalar functions on a two-dimensional plane with metric

$$ds^2=d\lambda^2/(\lambda^2-c^2)+d\mu^2/(1-\mu^2),$$

where c is a positive constant which fixes the manifold scale and $c<\lambda<\infty$, $-1<\mu<1$. The scalar Y is a twist potential and together with the four-dimensional metric component X uniquely determines the four-dimensional geometry for each possible black-hole exterior. As long as the scalars X and Y satisfy certain conditions on the orthogonally intersecting components of the boundary of the two-dimensional manifold, regularity of the four-dimensional geometry on the axisymmetry axis and horizon follows. These conditions are the following: As $\mu\to\pm1$, X and Y are well-behaved functions of λ and μ with

$$\begin{aligned}&X=O(1-\mu^2),\\&X^{-1}X_{,\mu}=-2\mu(1-\mu^2)^{-1}+O(1),\\&Y_{,\lambda}=O((1-\mu^2)^2),\quad Y_{,\mu}=O(1-\mu^2);\end{aligned}\tag{1}$$

as $\lambda\to c$, X and Y are well-behaved functions and

$$\begin{aligned}&X=O(1),\quad X^{-1}=O(1),\quad Y_{,\lambda}=O(1),\\&Y_{,\mu}=O(1).\end{aligned}\tag{2}$$

The absence of closed time-like lines in the space-time region exterior to the event horizon is ensured by requiring that X is greater than or equal to 0, with equality only on the axisymmetry axis, and asymptotic flatness is imposed by demanding that as $\lambda^{-1}\to0$, Y and $\lambda^{-2}X$ are well-behaved functions of λ^{-1} and μ with

$$\begin{aligned}&\lambda^{-2}X=(1-\mu^2)[1+O(\lambda^{-1})],\\&Y=2J\mu(3-\mu^2)+O(\lambda^{-1}),\end{aligned}\tag{3}$$

where J is the asymptotically conserved angular momentum.

The field equations are given by

$$E(X,Y)=F(X,Y)=0,\tag{4}$$

where

$$\begin{aligned}&E(X,Y)=\nabla\cdot(\rho X^{-2}\nabla X)+\rho X^{-3}(|\nabla X|^2+|\nabla Y|^2),\\&F(X,Y)=\nabla\cdot(\rho X^{-2}\nabla Y),\end{aligned}\tag{5}$$

$\rho=(\lambda^2-c^2)^{1/2}(1-\mu^2)^{1/2}$, and ∇ denotes the covariant derivative with respect to the two-dimensional metric above. Two well-behaved functions X and Y which satisfy Eqs. (4) and the boundary conditions uniquely determine (by quadratures) the components V, W, and U of the metric,

$$ds^2 = -V\,dt^2 + 2W\,d\varphi\,dt + X\,d\varphi^2 + U\left(\frac{d\lambda^2}{\lambda^2 - c^2} + \frac{d\mu^2}{1-\mu^2}\right),$$

of the vacuum region exterior to the event horizon. The ignorable coordinates have ranges $-\infty < t < \infty$ and $0 < \varphi < 2\pi$, and

$$(1-\mu^2)Y_{,\mu} = XW_{,\lambda} - WX_{,\lambda}; \quad -(\lambda^2 - c^2)Y_{,\lambda} = XW_{,\mu} - WX_{,\mu}.$$

It is a straightforward matter to deduce the following identity by using Eqs. (5):

$$\begin{aligned}(X_1X_2)^{-1}(Y_2-Y_1)[X_1{}^2F(X_1,Y_1) - X_2{}^2F(X_2,Y_2)] + \tfrac{1}{2}X_2{}^{-1}[(Y_2-Y_1)^2 + X_2{}^2 - X_1{}^2]E(X_1,Y_1)\\ + \tfrac{1}{2}X_1{}^{-1}[(Y_2-Y_1)^2 + X_1{}^2 - X_2{}^2]E(X_2,Y_2) + \tfrac{1}{2}\nabla\cdot\left[\rho\nabla\left(\frac{(X_2-X_1)^2 + (Y_2-Y_1)^2}{X_1X_2}\right)\right]\\ = \rho(2X_1X_2)^{-1}|X_1{}^{-1}(Y_2-Y_1)\nabla Y_1 - \nabla X_1 + X_2{}^{-1}X_1\nabla X_2|^2 + \rho(2X_1X_2)^{-1}|X_2{}^{-1}(Y_2-Y_1)\nabla Y_2 + \nabla X_2 - X_1{}^{-1}X_2\nabla X_1|^2\\ + \rho(4X_1X_2)^{-1}|(X_2+X_1)(X_2{}^{-1}\nabla Y_2 - X_1{}^{-1}\nabla Y_1) - (Y_2-Y_1)(X_2{}^{-1}\nabla X_2 + X_1{}^{-1}\nabla X_1)|^2\\ + \rho(4X_1X_2)^{-1}|(X_2-X_1)(X_1{}^{-1}\nabla Y_1 + X_2{}^{-1}\nabla Y_2) - (Y_2-Y_1)(X_1{}^{-1}\nabla X_1 + X_2{}^{-1}\nabla X_2)|^2. \quad (6)\end{aligned}$$

Now for a given pair of parameter values $c > 0$ and J there corresponds a Kerr solution with $|a| < m$ since the Kerr parameters a and m are related to the parameters c and J by $J = am$ and $c^2 = m^2 - a^2$. Suppose therefore that (X_1, Y_1) and (X_2, Y_2) correspond, respectively, to the Kerr solution $(|a| < m)$ and a second black-hole solution with the same parameter values c and J. The integration of Eq. (6) over the two-dimensional manifold and the application of Stokes theorem then leads to a boundary integral only, on the left-hand side of this equation. Application of the boundary conditions (1), (2), and (3) shows that the boundary integral vanishes and hence each of the nonnegative terms on the right-hand side of Eq. (6) must vanish. A simple manipulation of the resulting first-order partial differential equations leads to

$$\nabla[(Y_2 - Y_1)/X_1X_2] = 0,$$

and hence, with application of the boundary conditions once again, it follows that $Y_2 = Y_1$, and therefore $X_2 = X_1$, and the result is proven.

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