Renormalization-Group Calculation of Critical Exponents in Three Dimensions*

Geoffrey R. Golner and Eberhard K. Riedel
Department of Physics, Duke University, Durham, North Carolina 27706
(Received 16 December 1974)

A scaling-field representation of Wilson's exact renormalization-group equation is derived and used for the approximate calculation of critical exponents for the continuous-spin Ising model in three dimensions. The truncation of the hierarchy of scaling-field equations that retains only the four most relevant scaling fields yields the critical exponents $\nu \approx 0.60$, $\eta \approx 0.06$, and $\Delta_1 \approx -0.47$ (where Δ_1 denotes Wortis's correction-to-scaling exponent).

Within the renormalization-group approach,[1,2] the scaling theories for critical phenomena have been reformulated from an essentially microscopic standpoint.[3] The scaling-field idea[4,5] has played a prominent role in these developments, which grew out of Wilson's fixed-point hypothesis[1] and the concept of a spectrum of critical operators.[6,7] In several recent papers the scaling-field idea has been applied and developed further.[8-12] This Letter concerns a *microscopic derivation* of the basic scaling-field equations for Wilson's generalized Ising model[2] and the application of these equations to the calculation of critical exponents in three dimensions.

The method of calculation is based on an explicit scaling-field representation of Wilson's exact renormalization-group equation for the continuous-spin Ising model.[2] By an expansion about the Gaussian fixed point the Wilson equation is transformed into an infinite hierarchy of ordinary differential equations for scaling fields. The results are exhibited in Eqs. (4) and (9) below. The fixed-point properties of these equations determine critical exponents, and the flow properties of the equations yield scaling functions and crossover phenomena. Approximations are generated by truncating the hierarchy of equations according to the degree of relevance of the scaling fields to be retained. Preliminary results based on the two simplest, successive truncations of the scaling-field equations yield remarkably good estimates for the Ising critical exponents. The truncation retaining the two most relevant scaling fields yields $\nu \approx 0.55$ and $\eta = 0$. The truncation that retains the four most relevant fields yields improved values $\nu \approx 0.60$ and $\eta \approx 0.06$; and it yields $\Delta_1 \approx -0.47$ for Wortis's correction-to-scaling exponent.[7,13] Approximations to the full hierarchy of scaling-field equations introduce a weak dependence (to be discussed below) of critical exponents on redundant parameters in the renormalization-group approach.[14,15] The approach, if it converges, constitutes a systematic method for generating quantitative, improvable approximations to critical phenomena in *three dimensions*. [Previously, specific results for continuous-spin systems[2] were obtained near the molecular-field limit (by ϵ expansion about four dimensions) and near the spherical-model limit (by $1/n$ expansion).] The convergence of the method will be investigated by analyzing further truncations. (In the previous formal and semiphenomenological applications of the scaling-field idea the question of convergence could not be studied.) In summary, the potential significance of the scaling-field method is that it provides a *unified framework* for the calculation of critical exponents (including logarithmic corrections[5]), scaling functions,[8,12] and crossover phenomena.[8,12]

The method of calculation is described briefly. The exact Wilson equation [Eq. (11.17) of Ref. 2] determines the evolution of the effective renormalization-group Hamiltonian H_l as a function of l,

$$\frac{\partial H_l}{\partial l}=\int_q\left(\tfrac{1}{2}d\sigma_q+\vec{q}\cdot\nabla_q\sigma_q\right)\frac{\delta H_l}{\delta\sigma_q}+\int_q\left(\frac{d\rho}{dl}+2q^2\right)\left(\frac{\delta H_l}{\delta\sigma_q}\frac{\delta H_l}{\delta\sigma_{-q}}+\frac{\delta^2 H_l}{\delta\sigma_q\delta\sigma_{-q}}+\sigma_q\frac{\delta H_l}{\delta\sigma_q}\right). \tag{1}$$

The initial Hamiltonian, H_0, is the generalized, continuous-spin Ising Hamiltonian; spin variables are denoted by σ; the symbol $\int_q$ denotes the d-dimensional integral $\int d^dq$. The function $\rho(l)$ acts as a spin rescaling parameter that must be properly chosen in order to find a fixed point of the renormalization-group transformation.[2,15] If for a given fixed point H^* we write $d\rho^*/dl = 1+\Delta^*$ then the critical exponent η is determined by[2]

$$\eta = -2\Delta^*. \tag{2}$$

Associated with this fixed point is a set of translationally invariant eigendensities $O_i[xe^{-l};\sigma]$, determined by the renormalization-group equation linearized about H^*.[2]

The eigendensities provide an operator basis in the space of effective Hamiltonians H_l, which allows the expansion[4]

$$H_l = H^* + \textstyle\sum_i \mu_i(l) Q_i, \tag{3}$$

where the $Q_i = e^{-dl}\int_x O_i[xe^{-l};\sigma]$ are independent of l. Hence the evolution of H_l as a function of l can be equivalently described by the evolution of the coefficients $\mu_i(l)$. By substituting the expansion (3) into the Wilson equation (1) and projecting all resulting terms onto the basis Q_i via an operator-product expansion [Eq. (9) below] the Wilson equation is transformed into the infinite hierarchy of coupled differential equations,

$$\frac{d\mu_i}{dl} = y_i\mu_i + \sum_{j,k} a_{i,jk}\mu_j\mu_k + \Delta\left(\textstyle\sum_j a_{i,j}\mu_j + a_i\right), \tag{4}$$

where $\Delta = d\rho/dl - d\rho^*/dl$. The y_i are the eigenvalues associated with the basis operators Q_i.

The projection coefficients a are defined by

$$\int_q \left(\frac{d\rho}{dt} + 2q^2\right)\frac{\delta Q_j}{\delta\sigma_q}\frac{\delta Q_k}{\delta\sigma_{-q}} = \textstyle\sum_i a_{i,jk} Q_i, \tag{5a}$$

$$\int_q \left(2\frac{\delta H^*}{\delta\sigma_{-q}} + \sigma_q + \frac{\delta}{\delta\sigma_{-q}}\right)\frac{\delta Q_j}{\delta\sigma_q} = \sum_i a_{i,j} Q_i, \tag{5b}$$

$$\int_q \left(\frac{\delta H^*}{\delta\sigma_{-q}} + \sigma_q + \frac{\delta}{\delta\sigma_{-q}}\right)\frac{\delta H^*}{\delta\sigma_q} = \sum_i a_i Q_i. \tag{5c}$$

These equations are valid for any fixed-point Hamiltonian H^* and its associated set of basis operators Q_i.

For expansions (3) in terms of the basis of Gaussian eigenfunctionals, the projection coefficients a can be explicitly evaluated. The Gaussian solution of Eq. (1) has been discussed by Wilson and Kogut.[2] (In three dimensions it describes tricritical phenomena and in four dimensions critical phenomena in the mean-free limit.) The Gaussian fixed-point Hamiltonian is

$$H_G^* = -\tfrac{1}{2}\int_q u_2^*(q)\sigma_q\sigma_{-q} \tag{6}$$

with $u_2^*(q) = Aq^2/[Aq^2 + \exp(-2q^2)]$ and $d\rho^*/dl = 1$. The parameter A denotes the arbitrary normalization of the kinetic energy term in the Hamiltonian. The Gaussian eigenfunctionals have the general form

$$Q_{mp} = \sum_{i=1}^{m}\int_{q_1}\cdots\int_{q_i} v_{mp}^{(i)}(\vec{q}_1,\ldots,\vec{q}_i)\psi(q_1)\cdots\psi(q_i)\sigma_{q_1}\cdots\sigma_{q_i}, \tag{7}$$

and are labeled by the pair of indices $\{m,p\}$. In Eq. (7), the $v_{mp}^{(i)}$ are polynomials in $\vec{q}$ times a momentum conservation δ function, and $\psi(q) = \exp(-q^2)/[Aq^2 + \exp(-2q^2)]$. Under the linearized renormalization-group transformation, the l dependence of the eigenfunctional Q_{mp} is given by a multiplicative factor, $\exp(y_{mp}l)$ with $y_{mp} = d - \frac{1}{2}m(d-2) - p$.

Consider an arbitrary functional $P_0\{\sigma\}$. If it is expanded in terms of the Gaussian eigenfunctionals, $P_0\{\sigma\} = \sum_{m,p} a_{mp}Q_{mp}$, then, according to the *linearized* renormalization-group equation, it will evolve as

$$P_l\{\sigma\} = \sum_{m,p} a_{mp}\exp(y_{mp}l)Q_{mp}. \tag{8}$$

Thus, by decomposing the linearized evolution of $P_l\{\sigma\}$ in terms of the characteristic l dependence $\exp(y_{mp}l)$ of the Q_{mp}, the initial expansion coefficients a_{mp} can be calculated. The projection operator that accomplishes this decomposition of $P_l\{\sigma\}$ is derived by linearizing the integrated form of the exact renormalization-group equation [Eq. (11.7) of Ref. 2] about the Gaussian fixed point. Writing $H_l\{\sigma''\} = H_{G,l}^*\{\sigma''\} + P_l\{\sigma''\}$, where σ'' is the as yet *unrescaled* spin variable, this yields

$$P_l\{\sigma''\} = c\int_{\{s_q\}}\exp\left(-\tfrac{1}{2}\int_q s_q s_{-q}\right)P_0\left\{\left(\frac{1-\exp[-2\alpha_q(l)]}{B(q,l)}\right)^{1/2}s_q + \frac{\exp[-\alpha_q(l)]}{B(q,l)}\sigma_q''\right\}, \tag{9}$$

with $B(q,l) = u_2^*(q) + [1 - u_2^*(q)]\exp[-2\alpha_q(l)]$, the incomplete integration function $\alpha_q(l) = (e^{2l}-1)q^2 + \rho(l)$, and the normalization constant $c = \exp[-\frac{1}{2}\delta(0)\int_q \ln 2\pi]$. When $P_0\{\sigma\}$ is a polynomial in σ_q (as needed in our case), the functional integration over s_q is easily accomplished. After a final rescaling of the momentum and spin variables, $q' = qe^l$ and $\sigma_{q'}' = \exp(-\frac{1}{2}dl)\sigma_q''$, Eq. (8) is obtained. Thus, the projection operator (9) permits the unambiguous decomposition of the l dependence of any polynomial $P_0\{\sigma\}$ and hence the determination of the projection coefficients in Eq. (4). For example, the $p=0$ coefficients $a_{i0,j0,k0}$ of Eq. (5a) have the form $a_{i0,j0,k0}=c_{i,jk}I_n$, where the $c_{i,jk}$ are combinatorial factors, the subscript $n=\frac{1}{2}(k+j-i)-1$, $I_0=(1+\Delta)$, and, for $n>0$,

$$I_n=\int_{q_1}\cdots\int_{q_n}[(1+\Delta)+2(\vec{q}_1+\dots+\vec{q}_n)^2]\psi^2(\vec{q}_1+\dots+\vec{q}_n)\frac{\psi^2(q_1)}{u_2^*(q_1)}\cdots\frac{\psi^2(q_n)}{u_2^*(q_n)}. \tag{10}$$

Equations (4) and (9) are the central result of this paper. They define the scaling-field representation of the exact Wilson equation for the generalized, d-dimensional, continuous-spin Ising model. We use these equations as a starting point for the calculations of critical phenomena in *three dimensions* as described in the introduction.

The two successive truncations to the hierarchy of scaling-field equations investigated so far are, first, the coupled equations (denoted $E2$) for the two most relevant scaling fields μ_{20} and μ_{40}; and second, the coupled equations ($E4$) for the four most relevant scaling fields μ_{20}, μ_{40}, μ_{60}, and μ_{22}. For example, the truncation $E2$ is given by $d\mu_{20}/dl=2\mu_{20}+2I_0\mu_{20}^2+2I_1\mu_{20}\mu_{40}+I_2\mu_{40}^2$, and $d\mu_{40}/dl=\mu_{40}+8I_0\mu_{20}\mu_{40}+6I_1\mu_{40}^2$. Equations for the two and three most relevant *odd* scaling fields μ_{10}, μ_{30}, and μ_{50} were also considered and will be referred to as $O2$ and $O3$.

The solutions of the truncations $E2$ and $E4$ exhibit lines of non-Gaussian critical fixed points parametrized by the normalization constant A of the Gaussian fixed-point functional (6).[15] The associated critical exponents ν, which are determined via the standard eigenvalue method,[2] are shown in Fig. 1 as functions of A. First, for the two successive truncations the value of ν improves relative to the mean-field result $\nu=0.5$. For large A the two approximations yield $\nu\approx 0.55$ and $\nu\approx 0.60$, respectively. Second, over the interval $10^{-1}\leq A\leq 10^4$ the exponent ν varies by 11% for the first truncation ($E2$), but *only* by 1.7% for the second truncation ($E4$). (Exponents calculated from the untruncated equations should not depend on A at all.) Third, for both truncations the critical exponent ν becomes constant for $A\gtrsim 10^2$.

The determination of the critical exponent η is shown in Fig. 2. The truncated scaling-field equations exhibit critical fixed-point solutions for a range of values of the parameter Δ. Figure 2 shows for the truncation $E4$ the fixed-point coordinate μ_{22}^* as a function of Δ for six values of A. We determine the fixed-point value Δ^* by requiring that the non-Gaussian critical fixed point have the same normalization as the Gaussian fixed point of Eq. (6). That implies $\mu_{22}^*(\Delta^*)=0$, which determines the critical exponent η through Eq. (3). As A becomes large, the values for the exponent η converge towards $\eta\approx 0.062$ and become increasingly *independent* of the choice of the fixed-point condition for μ_{22}^*. For small A the value of η varies considerably as a function of A, and the fixed-point condition has no solution for $A\lesssim 10^{-2}$. For the truncation $E2$ no condition for Δ^* exists, and we choose $\Delta^*=0$ consistent with the Gaussian fixed-point value. The exponents ν plotted in Fig. 1 are those for the fixed

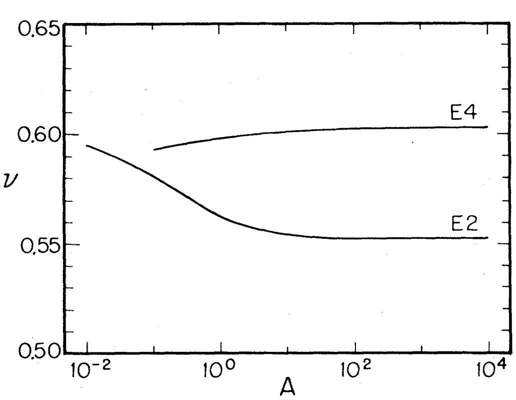

FIG. 1. Critical exponent ν as a function of A as obtained from the sets of two scaling-field equations ($E2$) and four scaling-field equations ($E4$).

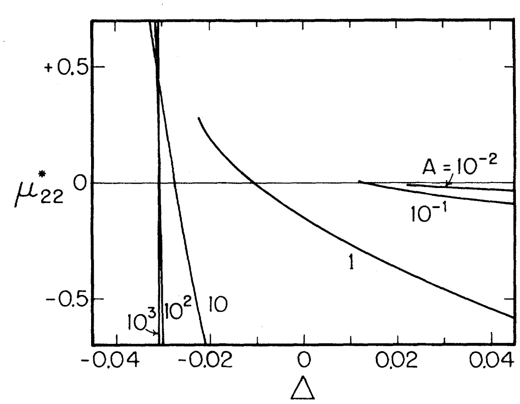

FIG. 2. Fixed-point coordinate μ_{22}^* as a function of Δ for six values of A. The lines of fixed points for $A=10^{-2}$, 10^{-1}, and 1 terminate as shown.

points with $\Delta = \Delta^*$ and $\Delta = 0$, respectively. The accuracy of the calculations presented here is to three significant figures.

The "odd" scaling-field equations allow an additional determination of η via the critical index $y_{10}^c = \frac{1}{2}(d + 2 - \eta)$. The truncations $O2$ and $O3$, when linearized about the critical fixed point of the "even" equations $E2$ and $E4$, respectively, reproduce exactly all input values of Δ. Thus the pairs of equations $[E2, O2]$ and $[E4, O3]$ constitute consistent truncations. The correction-to-scaling exponent Δ_1 is given by ν times the critical index[7] y_{40}^c; in approximation $E4$ it assumes, for large A, the value $\Delta_1 \approx -0.47$.

According to the theory[2,15] the critical exponents ν and η should be *independent* of the normalization parameter A, and the critical fixed-point solution should exist for only *one value* of Δ. The result of an ϵ-expansion calculation of η to order ϵ^2 from Eq. (1) agrees with both expectations.[16] Presumably the untruncated scaling-field expansion (if it converges) will also show this behavior. For the truncations studied in this paper we see that the ideal behavior is approached as A becomes large. For that reason we quote the values of the critical exponents as they are obtained for large A. The *a priori* significance of large A is not yet well understood. We expect that as the truncation is further removed the ideal "large-A" behavior will be approached for smaller values of A, with the results becoming universal in the untruncated limit.

We thank Professor K. G. Wilson for discussions and Professor H. Meyer for a critical reading of the manuscript.

*Work supported in part by the National Science Foundation through Grant No. GH-36882.

[1]K. G. Wilson, Phys. Rev. B 4, 3184 (1971).
[2]K. G. Wilson and J. Kogut, Phys. Rep. 12C, 75 (1974).
[3]For references to the original works on scaling, see M. E. Fisher, in *Proceedings of the Twenty-Fourth Nobel Symposium on Collective Properties of Physical Systems, Aspenaasgarden, Sweden, 1973*, edited by B. Lundqvist and S. Lundqvist (Academic, New York, 1973), p. 16.
[4]F. J. Wegner, Phys. Rev. B 5, 4529 (1972).
[5]F. J. Wegner and E. K. Riedel, Phys. Rev. B 7, 248 (1973).
[6]L. P. Kadanoff, Phys. Rev. Lett. 23, 1430 (1969).
[7]F. J. Wegner, Phys. Rev. B 6, 1891 (1972).
[8]E. K. Riedel and F. J. Wegner, Phys. Rev. B 9, 294 (1974).
[9]F. J. Wegner, J. Phys. C: Proc. Phys. Soc., London 7, 2109 (1974).
[10]D. R. Nelson and M. E. Fisher, to be published.
[11]F. J. Wegner, to be published.
[12]D. R. Nelson, to be published.
[13]M. Wortis, in Proceedings of the Newport Conference on Phase Transitions, 1970 (unpublished).
[14]F. J. Wegner, J. Phys. C: Proc. Phys. Soc., London 7, 2098 (1974).
[15]Compare also T. L. Bell and K. G. Wilson, Phys. Rev. B 10, 3935 (1974).
[16]P. Shukla and M. S. Green, Phys. Rev. Lett. 33, 1263 (1974); G. R. Golner and E. K. Riedel, Phys. Rev. Lett. 34, 171 (1975).

Charge-Separation Electric Fields in Laser Plasmas*

C. W. Mendel, Jr., and J. N. Olsen
Sandia Laboratories, Albuquerque, New Mexico 87115
(Received 15 January 1975)

Space-charge-separation electric fields have been measured in the expansion of laser plasmas generated by 40-psec, 1.06-μm laser pulses. The measurements were made at 2.5 and 5.0 mm from the carbon targets and on nanosecond time scales. Measured electric fields were as high as 1900 V/cm and exceed those expected from theory.

A major problem in laser plasmas has been that of making local, time-dependent measurements close to the target without disturbing the plasma. In the past, charged-particle probes have been used to measure plasma fields on long time scales.[1-3] Kalmykov, Timofeev, and Shevchuk[4] have used an ion beam to detect fields on a 10-nsec time scale. However, their scheme detected fields only when they reached predetermined values, thus calling for repeatable experiments for complete (in time) data. This Letter describes the first local measurement of space-charge-separation electric fields near a laser target (2.5 to 5 mm away), on a nanosecond time