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Spin Hydrodynamics of ^{3}He in the Anisotropic A Phase

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The linearized hydrodynamics of spin motion in A-^{3}He is derived from the broken symmetries and thermodynamics alone. Spin and diffusion modes are obtained without and with spin-orbit coupling and an external magnetic field.

The low-energy collective excitations of superfluid ^{3}He continue to receive great experimental and theoretical interest because of the fascinating interplay and richness of the phenomena encountered.[1] Five continuous symmetries are simultaneously broken[2-4] in A-^{3}He [gauge symmetry (1), rotation symmetry in real space (2) and in spin space (2)]. In the hydrodynamic limit one has to add an equal number of quasiconserved quantities[5-7] to the usual list of eight constants of the motion [mass (1), energy (1), momentum (3), spin (3)], which make up the hydrodynamics of the normal fluid. Consequently, A-^{3}He has probably the richest hydrodynamics of all systems known so far. In its various aspects it behaves simultaneously like an anisotropic superfluid,[5] a nematic liquid crystal,[7] or an antiferromagnet,[6] to mention only the more exotic among all the systems it can be compared with. In spite of its complexity this behavior in all its detail and generality comes as a direct consequence of the broken symmetries. In the impressive list of papers[1,2,8,9] which have already been addressed to the hydrodynamics of A-^{3}He, there appears to be none which makes this point in a complete way. Thus, the complete hydrodynamics of A-^{3}He has not yet been given.

In an earlier Letter[10] one of us has given the derivation of the orbital part of the hydrodynamics of A-^{3}He from three of the broken symmetries (gauge symmetry and spatial rotation symmetry) neglecting any coupling between the orbital and the spin hydrodynamics. In the present note we want to carry through the corresponding steps for the spin part. We will end up with a complete set of equations for all hydrodynamic modes of A-^{3}He. Only a linearized theory will be considered.

The spin hydrodynamics in normal ^{3}He is governed by the four conservation laws,

$$\partial\epsilon/\partial t+\nabla\cdot\vec{j}_\epsilon=0,$$
$$\partial m_i/\partial t+\nabla_k j_{ik}+\gamma\epsilon_{ijk}m_jH_k^{(e)}=0, \tag{1}$$

for the energy density ϵ and the magnetization $\vec{m}$ and their respective currents. $\vec{H}^{(e)}$ is an externally applied homogeneous magnetic field. γ is the gyromagnetic ratio. In A-^{3}He there exist two preferred directions which transform odd under time reversal and are thus related to certain angular momenta; one is in real space and is described by a unit vector $\vec{l}$, the other is in spin space and is described by a unit vector $\vec{n}$. Like for all vectors we have

$$(1/i\hbar)[L_i, l_j] = \epsilon_{ijk} l_k\,, \qquad (1/i\hbar)[S_i, n_j] = \epsilon_{ijk} n_k\,, \tag{2}$$

where $\vec{L}$ and $\vec{S}$ are the total orbital angular momentum and total spin, respectively. The commutators may be interpreted as Poisson brackets. In the following only such states will be considered which are infinitesimally close to spatially homogeneous states with $\vec{n} = \hat{n}^0$ and $\vec{l} = \hat{l}^0$ constant throughout the system.

The directions of $\hat{n}^0$ and $\hat{l}^0$ are completely independent as long as the small spin-orbit coupling due to magnetic dipole-dipole interaction is neglected, which we will do in the first part of this work. Thus, changes in the directions of $\vec{n}$ and $\vec{l}$ (not their absolute values) decay independently, and arbitrarily slowly in the long-wavelength limit, and have to be added to the list of hydrodynamic variables. Since the equations for $\vec{l}$ have been considered already,[10] we need only two additional equations of the form

$$(\hbar/2m)\dot{n}_i + Y_i = 0\,. \tag{3}$$

They are valid under the constraint $\dot{n}_i \hat{n}_i^0 = 0$ expressing the fact that $|\vec{n}|$ relaxes to its equilibrium value 1 in a microscopic time and does not change on the hydrodynamic time and length scale.

In the hydrodynamic limit thermodynamic equilibrium is established locally, i.e., we have the Gibbs relation

$$T\,d\rho s = d\epsilon - (h_i - H_i^{(e)})\,dm_i - (\hbar/2m)\psi_{ij}\,d(\nabla_j n_i)\,, \tag{4}$$

where again only the spin part has been written down. s is the entropy per unit mass; h_i and ψ_{ij} are the parameters which are thermodynamically conjugate to m_i and $\nabla_j n_i$, respectively. Only the gradients of $\vec{n}$ enter since a uniform twist of $\vec{n}$ is allowed by symmetry and cannot change the entropy or energy.

First, the reversible parts of the "currents" in Eqs. (1) and (2) are determined to lowest order in the gradients of the conjugate parameters T, h_i, ψ_{ij}. They have to satisfy the requirements of zero entropy production, reversibility, and covariance under independent rotations of real space and spin space. Dropping all surface terms, we find

$$\vec{j}_\epsilon^{\,R} = 0; \quad \vec{j}_{ik}^{\,R} = \gamma' \epsilon_{ipj} \hat{n}_p^0 \psi_{jk}; \quad Y_i^R = -\gamma \epsilon_{ipk} \hat{n}_p^0 (h_k - H_k^{(e)})\,. \tag{5}$$

The last equation describes the linear response of $\vec{n}$ to an applied magnetic field $\vec{H}^{(e)}$. By the commutation relation (2) the change of $\vec{n}$ in a homogeneous field $\vec{H}^{(e)}$ is given exactly by $\dot{\vec{n}} = \gamma \vec{H}^{(e)} \times \vec{n}$. Comparison with Eq. (5) yields $\gamma' = (\hbar/2m)\gamma$.[12]

Next, we calculate the dissipative parts of the currents, again to lowest order in the gradients. They have to satisfy irreversibility, covariance under independent rotations of real space and spin space, and positivity of the entropy production. Furthermore we impose the Onsager symmetry principle on the matrix of coefficients connecting the fluxes $\vec{j}_\epsilon^{\,D}$, $\dot{j}_{ik}^{\,D}$, and Y_i^D with the thermodynamic forces. The result is

$$j_{\epsilon i}^{\,D} = -\kappa_{ij}' \hat{n}_k^0 \nabla_j h_k\,, \quad Y_i^D = -\nu \nabla_j \psi_{ij}\,, \quad j_{ik}^{\,D} = -\mu_{iklm} \nabla_m h_l - \hat{n}_i^0 (\kappa_{km}'/T) \nabla_m T\,, \tag{6}$$

with

$$\kappa_{ij}' = \kappa_\perp'(\delta_{ij} - \hat{l}_i^0 \hat{l}_j^0) + \kappa_\parallel' \hat{l}_i^0 \hat{l}_j^0\,,$$
$$\mu_{iklm} = (\delta_{km} - \hat{l}_k^0 \hat{l}_m^0)[\mu_1(\delta_{il} - \hat{n}_i^0 \hat{n}_l^0) + \mu_2 \hat{n}_i^0 \hat{n}_l^0] + \hat{l}_k^0 \hat{l}_m^0 [\mu_3(\delta_{il} - \hat{n}_i^0 \hat{n}_l^0) + \mu_4 \hat{n}_i^0 \hat{n}_l^0]\,. \tag{7}$$

The normal heat conduction involving the transport parameters $\kappa_\parallel$, $\kappa_\perp$ has already been included in the orbital part[10] and is omitted here. The positivity of entropy production implies

$$\nu > 0, \quad \mu_i > 0, \quad \kappa_\parallel > 0, \quad \kappa_\perp > 0,$$
$$\kappa_\parallel \mu_4 > \kappa_\parallel'/T\,, \quad \kappa_\perp \mu_2 > \kappa_\perp'/T$$

for the new transport coefficients.

Our equations are now complete if we are able to connect the conjugate parameters with m_i, $\nabla_i n_k$ by an equation of state. The latter can be taken in linearized form, and must be reversible and covariant under space and spin rotation. Hence,

$$h_i = \chi_{ij}^{-1} m_j\,,$$
$$\psi_{ij} = (\hbar/2m) M_{jl} (\delta_{ik} - \hat{n}_i^0 \hat{n}_k^0) \nabla_l n_k\,, \tag{8}$$

with

$$\chi_{ij}^{-1}=\chi_\perp^{-1}(\delta_{ij}-\hat{n}_i^0\hat{n}_j^0)+\chi_\parallel^{-1}\hat{n}_i^0\hat{n}_j^0,$$
$$M_{ij}=M_\perp(\delta_{ij}-\hat{l}_i^0\hat{l}_j^0)+M_\parallel\hat{l}_i^0\hat{l}_j^0. \qquad (9)$$

The two mass densities $M_\parallel$, $M_\perp$ and the two susceptibilities $\chi_\parallel$, $\chi_\perp$ are positive if the system is stable.

Our equations are now complete. Their reversible part is in agreement with results by Combescot,[9] derived from a Landau quasiparticle description. Most closely related to our procedure and results is the work of Halperin and Hohenberg[6] on spin waves in antiferromagnets. In fact, if we neglect the additional complication due to the preferred spatial direction $\vec{l}$, our equations go over into theirs; A-^{3}He and antiferromagnets have a spin order parameter with a similar structure. It is easy to solve the equations for their normal modes. For zero external field we obtain a diffusion mode, for $(\hat{n}^0\cdot\vec{m})$, with the frequency

$$\omega_1=iBk^2, \qquad (10)$$

where

$$B=(\mu_2k_\perp^2+\mu_4k_\parallel^2)/\chi_\parallel k^2; \qquad (11)$$

B is a direction-dependent spin-diffusion constant. In spite of the cross coupling between $(\hat{n}^0\cdot\vec{m})$ and ϵ, introduced in Eq. (6), second sound and the diffusion of $(\hat{n}^0\cdot\vec{m})$ are completely decoupled in lowest order of k. The components of $\vec{m}$ and $\vec{n}$ which are transverse to $\hat{n}^0$ propagate as spin waves with the frequencies

$$\omega_{3,4}=\omega_{5,6}=\pm ck+\tfrac{1}{2}i(D_1+D_2)k^2 \qquad (12)$$

containing the direction-dependent phase velocity

$$c=(\hbar\gamma/2m)[(M_\perp k_\perp^2+M_\parallel k_\parallel^2)/\chi_\perp k^2]^{1/2} \qquad (13)$$

and the direction-dependent diffusion constants

$$D_1=\nu(M_\perp k_\perp^2+M_\parallel k_\parallel^2)/k^2,$$
$$D_2=\chi_\perp^{-1}(\mu_1k_\perp^2+\mu_3k_\parallel^2)/k^2. \qquad (14)$$

The quite different origin of the anisotropy of the two latter parameters should be noted.

Let us now take into account the small spin-orbit coupling in ^{3}He, arising from the magnetic dipole-dipole interaction between the ^{3}He atoms, whose importance for the superfluid phases was first demonstrated by Leggett.[2] It contributes to the internal energy a term which is dependent on the relative direction of $\vec{n}$ and $\vec{l}$ and does not vanish even in the limit $k\to 0$.

From now on we will use the same reference system for spin vectors and orbit vectors. The spin-orbit energy $\epsilon_{\text{s.o.}}$ can be expanded around its local minimum, which, in the A phase, is known to be at $\vec{n}=\vec{l}$.[1-4] For small deviations we have

$$\epsilon_{\text{s.o.}}=(\hbar/2m)^2\tfrac{1}{2}b(\vec{n}-\vec{l})^2, \quad b>0, \qquad (15)$$

where uniaxial symmetry has been used again. Introducing this additional energy into the Gibbs relation (4) we find that our previous calculations go through essentially unaltered, except for the replacements

$$\nabla_j\psi_{ij}\to\nabla_j\psi_{ij}-(\hbar/2m)b(n_i-l_i),$$
$$\nabla_j\varphi_{ij}\to\nabla_j\varphi_{ij}+(\hbar/2m)b(n_i-l_i). \qquad (16)$$

They lead to the following spin-orbit contributions to the equations of motion:

$$\dot{m}_i+\nabla_kj_{ik}=(\hbar/2m)^2\gamma b\epsilon_{ijk}\hat{n}_j^0(\vec{n}_k-\vec{l}_k),$$
$$(\hbar/2m)\dot{n}_i+Y_i=-(\hbar/2m)b\nu(\vec{n}_i-\vec{l}_i). \qquad (17)$$

Corresponding contributions have to be added to the equations of Ref. 10, which we summarize in a footnote.[12] In order to solve the equations, we make use of the empirical fact that $\vec{l}$ changes on a time scale which is much longer than the time scales set by the spin-wave frequencies (cf. Ref. 2). Then the instantaneous value of $\vec{l}$ defines a reference frame in which $\vec{n}$ orients itself, i.e., we may put $\hat{n}^0=\vec{l}$. The orbit waves are now decoupled from the spin waves and we may solve for the latter. The results can be generated from Eq. (12) by the replacements

$$ck^2\to c^2k^2+\chi_\perp^{-1}b\gamma\hbar/2m,$$
$$D_1k^2\to D_1k^2+\nu b. \qquad (18)$$

Thus, the spin waves are no longer true Goldstone modes of the system. They are pushed to a nonzero frequency and a finite lifetime for $k=0$, because of the spin-orbit symmetry violation of the dipole-dipole forces.

For $\vec{H}^{(e)}\neq 0$, equilibrium is established if $\vec{m}=\overleftrightarrow{\chi}\cdot\vec{H}^{(e)}\parallel\vec{H}^{(e)}$ which requires either $\hat{n}^0\parallel\vec{H}^{(e)}$ or $\hat{n}^0\perp\vec{H}^{(e)}$. In A-^{3}He we have[3,4] $\chi_\parallel<\chi_\perp$ and only the latter configuration is stable. The normal-mode spectrum in the case takes the form

$$\omega_1=ik^2\frac{Bc^2k^2+D_1\gamma^2H^{(e)2}}{c^2k^2+\gamma^2H^{(e)2}} \qquad (19)$$

for the diffusion mode, where spin-orbit coupling is omitted. The spin-wave spectrum of the longitudinal (parallel to $\vec{H}^{(e)}$) components of $\vec{m}$, $\vec{n}$ are decoupled from the other modes. Their frequencies $\omega_{3,4}$ are given in Eq. (12). The spin-wave spectrum of the components which are transverse to $\vec{H}^{(e)}$ and $\hat{n}^0$ is given by

$$\omega_{5,6} = \pm (c^2k^2 + \gamma^2 H^{(e)2}) + \frac{i}{2}k^2\left(D_2 + \frac{D_1c^2k^2 + B\gamma^2H^{(e)2}}{c^2k^2+\gamma^2H^{(e)2}}\right). \tag{20}$$

Spin-orbit coupling effects again are simply taken into account by the replacements $c^2k^2 \to c^2k^2 + \gamma\chi_\perp^{-1} \times b\hbar/2m$, $D_1k^2 \to D_1k^2 + \nu b$ leading to the frequency shifts observed in NMR experiments[13] and explained or predicted by Leggett.[2] Our results thus reproduce Leggett's results[2] for $k=0$, include the damping by spin-orbit coupling (cf. Combescot and Ebisawa[14]) and describe spin-wave effects which occur for $k \neq 0$.

Let us finally remark that spin motion and orbit motion are really coupled in A-^{3}He. Thus, the Goldstone mode which remains after switching on the spin-orbit interaction is not of a completely orbital nature, as in the approximation we employed, but contains spin admixtures as well. The latter will be small, however, because of the quite different time scales involved. A more detailed account of our results will be published elsewhere.

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[12]First,

$$\sigma_{ik} = \sigma_{ik}^0 + b[\alpha_1 \hat{l}_i^0(n_k - l_k) + \alpha_2 \hat{l}_k^0(n_i - l_i) - \xi_{ikj}(n_j - l_j)],$$

where σ_{ik}^0 is the stress tensor, omitting the spin-orbit coupling energy. σ_{ik} is no longer symmetric, i.e., $\vec{L}$ is no longer conserved. Second,

$$(\hbar/2m)\dot{\hat{l}}_i + X_i = (\hbar/2m)b[\beta\epsilon_{ijk}\hat{l}_k^0(n_j - l_j) + \eta(\delta_{ik} - \hat{l}_i^0\hat{l}_k^0)(n_k - l_k)].$$

The $\hat{l}$ of Ref. 10 is replaced by $\hat{l}^0$ in the notation employed here.

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